JOURNAL OF FUNCTIONAL ANALYSIS 3, 435–494 (1969)

Characters of Algebraic Solvable Groups*

L. PUKANSZKY

Department of Mathematics,
University of Pennsylvania, Philadelphia, Pennsylvania 19104

Communicated by Irving Segal

Received June 24, 1968

1.0. Let G be a connected and simply connected solvable Lie group with the Lie algebra $\mathscr{L}$. One says, that G is of type I, if any factor on a separable Hilbert space, generated by the operators of a continuous unitary representation of G, is of type I. It has been shown recently by L. Auslander and B. Kostant (cf. [*1*]), that the set of equivalence classes of irreducible unitary representations of G, if of type I, can be described by aid of the orbits of the representation, which is contragredient to the adjoint representation of G and acts on the dual of the underlying space of $\mathscr{L}$; this representation will be referred to as the coadjoint representations of G in the sequel. To each orbit of the sort just mentioned there corresponds a family of equivalence classes, the members of which can be parametrized by the points of a torus, having as dimension the first Betti number of the orbit under consideration (cf. Section 3 below for further details). One says, that G is exponential, if the exponential map establishes an analytic isomorphism between the underlying space of $\mathscr{L}$ and G respectively. If G is exponential, the orbits of the coadjoint representation are simply connected, and thus we have a bijection between the set of equivalence classes of irreducible unitary representations and the family of all orbits of the coadjoint representation. The existence of this bijection was established first by A.A. Kirillov in the nilpotent case, and extended later to the general exponential case by P. Bernat.

Let us assume now, that G is nilpotent, and let T be an irreducible unitary representation, belonging to the orbit O, of G. It was shown by Kirillov, that if φ is a C^∞ function having a compact support on G. and if we set

$$T(\varphi) = \int_G \varphi(a)\, T(a)\, da$$

* This research was supported by NSF GP 7683.

where da is a Haar measure on G, then $T(\varphi)$ is a trace class operator, the trace of which can be obtained as follows. We write $\varphi(l)$ $(l \in \mathscr{L})$ for the function corresponding to φ on $\mathscr{L}$ via the exponential map, and fixing a positive translation invariant measure dl on $\mathscr{L}$ we set

$$\hat{\varphi}(l') = \int_{\mathscr{L}} \varphi(l)\, e^{i(l,l')}\, dl, \qquad (l' \in \mathscr{L}').$$

Then we have

$$\operatorname{Tr}(T(\varphi)) = \int_{O} \hat{\varphi}(l')\, dv \tag{1}$$

where dv is a positive invariant measure (called the canonical measure) on O, and the integral on the right converges absolutely. This relation can be used to pair off orbits and irreducible representations (cf. for all this (*15*], Deuxième Partie, Chapitres II–III).

If G is exponential, but not necessarily nilpotent, an irreducible representation can be CCR (that is $T(\varphi)$ is absolutely continuous for any $\varphi \in C_c^\infty$), only if the corresponding orbit is closed (cf. [*17*], Theorem 1). We showed in a previous paper, that if T is such a representation, and if G satisfies the further condition, that the image of its Lie algebra in the adjoint representation is algebraic, then for any positive definite C_c^∞ function φ a formula similar to (1) holds. The sole difference is, that prior to forming the Fourier transform of $\varphi(l)$ one has to multiply it with an expression formed of the roots of $\mathscr{L}$ (cf. [*17*], Theorem 2). We recall, that a root of $\mathscr{L}$ is a complex-valued linear form on $\mathscr{L}$, corresponding to some simple quotient representation of the adjoint representation. In particular, if $\varphi \in C_c^\infty$, the operator $T(\varphi)$ is of class Hilbert–Schmidt.

The purpose of the present paper is to extend the latter result to any connected and simply connected solvable group with a Lie algebra $\mathscr{L}$, the image of which in the adjoint representation is algebraic. By virtue of a theorem of M. Goto this is the case if and only if $\mathscr{L}$ is isomorphic to an algebraic Lie algebra (cf. [*11*], Theorem 5, p. 41). Equivalently, a Lie group belongs to the class to be considered here, if and only if it is isomorphic to the universal covering group of the connected component of a real algebraic solvable group. It was proved by J. Dixmier, that any algebraic group, or its connected component is of type I (cf. [*8*], Theorem 1, p. 326). By a modification of his argument we prove first, that the same situation holds true for any group of the class described above (cf. Section 2 below), and therefore the theory of Auslander and Kostant can be applied to the classification of its irreducible representations. Next we consider an irreducible representation T corresponding to a closed orbit O and show, that if φ is of class C_c^∞, the operator $T(\varphi)$ is of class Hilbert–Schmidt, and if φ is in addition positive definite and its support lies in a fixed open set, containing the identity and not depending on the choice of T, we derive a formula for the trace of $T(\varphi)$ (cf. below for the precise statement). In particular, it turns out, that this trace is the same for any two representations belonging to the same orbit.

In deriving these results, when compared with the exponential case, new phenomena have to be dealt with. First, since the exponential map, in general, establishes an analytic isomorphism only between a variety, arising out of the underlying space of the Lie algebra by removing a countable sequence of hyperplanes, and its image (cf. [*9*], Théorème 2), to obtain an analytic expression, analogous to (1), for the trace, we have to impose the restriction, indicated above, on the support of φ. Next, the previous discussions of the trace formula rely more or less directly on the fact, that in the exponential case any irreducible representation can be obtained by taking the representation induced by a one-dimensional representation of an appropriately chosen connected subgroup, making it possible to reduce the computation of the trace to that of integral operators. This circumstance in the general case, however, retains its validity only if classical induction is replaced by holomorphic induction as in [*1*] and [*10*]. The main consequence of all this for us is, that by adopting a suitable extension of the inductive procedure employed by Kirillov for the derivation of the trace formula in the nilpotent case, we shall only have reduced the general problem to that of computing the trace for a special class of groups encountered, in the context of representations of algebraic groups, first by Dixmier (cf. [*8*] especially the proof of Theorem 1, p. 326; cf also [*10*], 5.7–5.12). In our case these are going to be groups, the Lie algebra of which is the extension, obtained by considering an abelian algebraic algebra of semi-simple derivations with purely imaginary roots, of the nilpotent algebra generated by the elements $\{I, p_i, q_j;\ \ i, j = 1, 2,\ldots, n\}$ satisfying the commutation relations $[p_i, q_j] = \delta_{ij} I$. These will be called special groups in the sequel. Here additional considerations, based on the realization, discovered by V. A. Bargmann and I. E. Segal, in a Hilbert space of holomorphic functions for the infinite dimensional irreducible unitary representations of the nilpotent special groups will have to be applied (cf. Section 6). The proof of the convergence of the integrals, occurring in the trace formula, as in [*17*], utilizes results of L. Hörmander on the asymptotic behaviour of polynomials in several real variables.

The final formula can be stated in the following fashion (cf. also 5.1).

Let G be a group of our class belonging to the Lie algebra $\mathscr{L}$. We denote by $\mathscr{F}$ the collection of all roots of $\mathscr{L}$, and by B the open connected subset of $\mathscr{L}$, given by $\{l; \operatorname{Im} \alpha(l) < 2\pi, \alpha \in \mathscr{F}\}$. The restriction of the exponential map to B is an analytic isomorphism with its image. Let φ be a C_c^∞ positive definite function, the support of which is contained in $\exp B$. Then we have

$$\operatorname{Tr}(T(\varphi)) = \int_O \hat{\omega}(l')\, dv.$$

Here $\hat{\omega}(l')$ is the Fourier transform, over $\mathscr{L}$, of a function ω, obtained by multipying the function, corresponding to φ on $\mathscr{L}$ via the exponential map, by a factor of the following form

$$(\Delta(\exp l))^{1/2} \prod_{\alpha \in \mathscr{F}_1} \frac{\exp(\frac{1}{2}\alpha(l)) - \exp(-\frac{1}{2}\alpha(l))}{\alpha(l)} \prod_{\beta \in \mathscr{F}_2} \frac{\exp \beta(l) - 1}{\beta(l)}$$

where $\mathscr{F}_1$ and $\mathscr{F}_2$ are disjoint subfamilies of $\mathscr{F}$, and $d(a_0 a) = \Delta(a_0)\, da$, if da is the right invariant measure on G used in forming $T(\varphi)$. Finally dv is a positive invariant measure on O, which can be obtained by precisely the same algorithm, as in the nilpotent case (cf. [*16*] and 5.6, 6.7 below). The integral on the right hand side converges absolutely. The families of roots $\mathscr{F}_1$ and $\mathscr{F}_2$ depend on O only, and not on the particular choice of the representation T corresponding to the closed orbit O. If G is exponential, $\mathscr{F}_2$ is empty, and we reobtain the formula of Theorem 2 in [*17*]. If, on the other hand, G is a special group, $\mathscr{F}_1$ is empty and we have also $\Delta \equiv 1$ (cf. 6.5).

The content of the present paper is as follows. In Section 2 below we prove, that solvable groups, for which $ad\mathscr{L}$ is algebraic, are of type I. In Section 3 we explain some basic facts concerning the holomorphic induction. In Section 4 irreducible representations with closed orbits are connected with representations of, not necessarily connected, closed subgroups, and in Section 5 a relation between their characters will be established. This will reduce the verification of the final formula to the computation of the trace in the case of a special group; this will be done in Section 6.

The reader is assumed to be familiar with the standard facts of the theory of induced representations (cf. [*12*] and [*2*] Chapter I, Sections 9–10).

2.1. All the Lie algebras and groups to be considered in this paper are assumed to be solvable, and unless stated otherwise, defined over the reals.

a) Let V be a finite dimensional real vector space; we denote by $L(V)$ the collection of all endomorphisms of V. Let $\mathscr{L}$ be a (not necessarily solvable) subalgebra of $L(V)$; we recall, that $\mathscr{L}$ is called algebraic, if with $a \in \mathscr{L}$ any replica of a belongs to $\mathscr{L}$ (cf. [*4*] and [*6*]). If $\mathscr{L}$ is solvable, it is algebraic if and only if, denoting by μ the ideal composed of all nilpotent elements of $\mathscr{L}$, there exist an algebraic abelian subalgebra $\mathfrak{a}$ of semi-simple endomorphisms, of $\mathscr{L}$, such that $\mathscr{L} = \mathfrak{n} + \mathfrak{a}$ holds. An abelian algebra of semi-simple endomorphisms is algebraic, if and only if its complexification possesses a base, the elements of which have integral eigenvalues only (cf. for all this [*13*], Corollary of Theorem 1, Theorem 2 and Theorem 4). Also, $\mathscr{L}$ is algebraic if and only if the connected analytic subgroup G, belonging to $\mathscr{L}$, of $GL(V)$ is the connected component of the identity of an algebraic subgroup $\mathfrak{G}$ of $GL(V)$. This statement, pointed out for the complex case in [*6*], can be shown for the reals as follows. Let $\mathfrak{G}$ be as above and $\mathscr{L}$ its Lie algebra; then $\mathscr{L}$ is algebraic. In fact, let a be an element in $\mathscr{L}$; we denote by s and n its semi-simple and nilpotent component resp. If $\mathscr{P}(A)$ is a polynomial in the matrix coefficients, taken with respect to some base in V, of $A \in L(V)$, and if $\mathscr{P}(\mathrm{Exp}(ta)) \equiv 0$ for all real t, one shows easily, that also $\mathscr{P}(\exp(ts)) \equiv 0$, and therefore s, and thus also n belong to $\mathscr{L}$. Next one observes, that a' is a replica of a if and only if $a' = s' + c \cdot n$, where c is a constant, and s' a replica of s. Thus it suffices to establish, that we have also $\mathscr{P}(\exp(ts')) \equiv 0$, which can easily be done by recalling, that there exist a base $\{f_k;\ 1 \leqslant k \leqslant n\}$ in the complexification of V, such that $sf_k = \lambda_k f_k$, $s'f_k = \mu_k f_k$ for all k, and if $\sum_{k=1}^{n} m_k\lambda_k = 0$, where the m_k's are integers, then we have also $\sum_{k=1}^{n} m_k\mu_k = 0$. Conversely, if $\mathscr{L}$ is algebraic, and if we write $\mathscr{L} = \mathfrak{n} + \mathfrak{a}$ as above, then the connected subgroup N, belonging to $\mathfrak{n}$, of $GL(V)$ is algebraic. Furthermore, by taking into account the characterization, given above, of algebraic abelian algebras of semi-simple endomorphisms, one easily determines an algebraic group with the Lie algebra $\mathfrak{a}$. Finally, to obtain an algebraic group for $\mathscr{L}$, one can, for instance, apply Théorème 14, p. 175 in [*5*]. In particular, the definitions of an algebraic Lie algebra given in [*6*] and [*5*] (p. 171) are equivalent in our case.

b) Let $\mathscr{L}$ be a real solvable Lie algebra; we write $\mathscr{L} = (a)$ if and only if $\mathscr{L}$ has a faithful linear representation λ on a finite dimensional real vector space, such that $\lambda(\mathscr{L})$ is algebraic. By virtue of Theorem 5 in [*11*], for $\mathscr{L} = (a)$ it is necessary and sufficient, that the image of $\mathscr{L}$ through the adjoint representation be algebraic. Also, $\mathscr{L} = (a)$ if and only if $\mathscr{L} = \mathfrak{n} + \mathfrak{a}$, where $\mathfrak{n}$ is the greatest nilpotent ideal of $\mathscr{L}$, and $\mathfrak{a}$ is an abelian subalgebra, such that $ad\mathfrak{a}$ is algebraic and consists of semi-simple endomorphisms; on the right hand side we have direct sum of subspaces.

c) Let G be a connected and simply connected solvable group with the Lie algebra $\mathscr{L}$; we shall write $G = (a)$ if $\mathscr{L} = (a)$.

2.2. The proof of the following proposition is an adaptation of the reasonings in [*8*].

PROPOSITION. *Let G be a connected and simply connected solvable Lie group with the Lie algebra $\mathscr{L}$, such that $ad\mathscr{L}$ is algebraic. Then G is of type* I.

Proof. We are going to proceed by induction, assuming the statement to be true for groups with a dimension less than that of G. Since for dim $G = 1$ there is nothing to be proved, below we assume dim $G > 1$, and distinguish the following subcases.

a) Let us assume first, that *T is a factorial representation of G such, that its kernel is not discrete.* Let J be a nonzero ideal of $\mathscr{L}$ such that $T \mid \exp J \equiv I$; here $\exp J$ stands for the connected subgroup, belonging to J, of G. Let us write $\tilde{G} = G/\exp J$. Since T arises by lifting a factorial representation of $\tilde{G}$ up to G, to prove, that T generates a factor of type I, by virtue of our inductive assumption it suffices to show, that $\tilde{G} = (a)$, or, that $\tilde{\mathscr{L}} = \mathscr{L}/J = (a)$. To this end let us observe, that if $\mathscr{B}$ is an algebraic subalgebra of $L(V)$, and if W is a subspace, invariant with respect to $\mathscr{B}$, of V, then the subalgebra $\mathscr{B}_1$, arising from $\mathscr{B}$ by taking quotients, of $L(V/W)$ is also algebraic. Therefore, to obtain the desired conclusion it is enough to note, that by writing $V = \mathscr{L}$, $W = J$ and $\mathscr{B} = \operatorname{ad} \mathscr{L}$ we get $\mathscr{B}_1 = \operatorname{ad} \tilde{\mathscr{L}}$, and thus $\tilde{\mathscr{L}} = (a)$.

b) Next we suppose, that the kernel of the factorial representation T is discrete. Let J be an abelian ideal of $\mathscr{L}$; J can be identified with $\exp J$ via the exponential map, and therefore the dual of $\exp J$ identifies with J'. G acts on J by inner automorphisms; we obtain the dual of this action on J' by considering the representation $\rho(a) = (Ad(a^{-1}))'$ of G on $\mathscr{L}'(a \in G)$, and by taking its quotient on $J' = \mathscr{L}'/J^{\perp}$, where $J^{\perp}$ is the annihilator of J in $\mathscr{L}'$. Since we have $\mathscr{L} = (a)$, the set $(\operatorname{ad} \mathscr{L})'$ of the transposed of all elements in $\operatorname{ad} \mathscr{L}$ is algebraic in $L(\mathscr{L}')$, and therefore we can conclude as in a) above, that $d\tau(\mathscr{L})$ is algebraic in $L(J')$. This implies that the orbits of τ on J' are those of the connected component of the identity in an algebraic subgroup of $GL(J')$, and in this fashion J'/τ is countably separated (cf. for this the theorem of C. Chevalley, quoted in [8], p. 316). From this we conclude, that the spectral measure corresponding to $T \mid \exp J$ is concentrated on one single orbit O. *Let us assume now, that J can be chosen such, that* $\dim O > 0$. Let p be a fixed element in O, S its stabilizer in G, and S_c the connected component of the identity in S. In what follows we shall prove, that $S_c = (a)$, and that there exist a discrete subgroup Γ in the center of G, such that ΓS_c is closed and has a finite index in S. Assuming this for a moment, we finish proving, that T generates a factor of type I, as follows. First we recall, that there exist a factorial representation U, inducing T in G, of S, and the types of the factors generated by T and U are identical; hence it suffices to know, that S is of type I. But because of $\dim O > 0$, the codimension of S_c in G is positive, and since $S_c = (a)$, by virtue of the assumption of our inductive procedure, S_c is of type I. Since Γ is in the center of G, this implies, that ΓS_c, too, is of type I. Finally, since the index of ΓS_c in S is finite, we can conclude, that S is of type I (cf. [8], Lemma 3, p. 319).

Since we have $\mathscr{L} = (a)$, we can assume, that $\mathscr{L}$ is an algebraic subalgebra of $L(V)$, where V is some finite dimensional real vector space. We denote by $\bar{G}$ the closed analytic subgroup, determined by $\mathscr{L}$, of $GL(V)$, and write ϕ for the canonical homomorphism of G onto $\bar{G}$. G is the connected component of an algebraic group $\mathfrak{G}$, of which we can assume, that $aJa^{-1} \subseteq J$ for all a in $\mathfrak{G}$. Let us identify $L(V)$ with its dual by means of the nondegenerate bilinear form $B(a, b) = \mathrm{Tr}(ab)$ $(a, b \in L(V))$. Denoting by $J_B{}^{\perp}$ the orthogonal complement of J with respect to B, we have $J' = L(V)/J_B{}^{\perp}$. Fixing an element $\bar{p}$, lying over $p \in O \subset J'$, in $L(V)$ we form the algebraic group $\bar{S}_1 = \{a;\ a \in \mathfrak{G},\ a\bar{p}a^{-1} \in J_B{}^{\perp}\}$, and write $\bar{S}$ for its intersection with $\bar{G}$. S is the complete inverse image, through ϕ, of $\bar{S}$, and also $\phi(S_c) = \bar{S}_c$; therefore we have $S_c = (\mathrm{a})$. Let us write Γ for the kernel of ϕ; Γ is a discrete central subgroup of G. ΓS_c is the complete inverse image of $\bar{S}_c$, thus ΓS_c is a closed invariant subgroup of S. Finally we observe, that $S/\Gamma S_c$ is finite, since so is evidently the index of $\bar{S}_c$ in $\bar{S}$.

c) Let us assume finally, that *given a factorial representation T of G, no ideal, satisfying the conditions described in* a) *or* b) *above, can be found in $\mathscr{L}$*. Then the connected subgroup N, belonging to the nilradical $\mathfrak{n}$ of $\mathscr{L}$, of G has the structure of a special group (cf. 1.). In fact, denoting by C the center of $\mathfrak{n}$, we claim, that $\dim C = 1$, and that $[\mathfrak{n}, \mathfrak{n}] = C$ (cf. [8], Lemma 10). The first assertion is evident, since our assumption implies, that $\mathscr{L}$ has no abelian ideal of a dimension greater than 1. If $\dim[\mathfrak{n}, \mathfrak{n}] > 1$, we can find two ideals $\mathfrak{n}_1$ and $\mathfrak{n}_2$ of $\mathfrak{n}$, such that dim $\mathfrak{n}_j = j$, and that $\mathfrak{n}_j \subset [\mathfrak{n}, \mathfrak{n}]$ $(j = 1, 2)$. But then we have $[[\mathfrak{n}, \mathfrak{n}], \mathfrak{n}_2] = [[\mathfrak{n}, \mathfrak{n}_2], \mathfrak{n}] \subseteq [\mathfrak{n}, \mathfrak{n}_1] = 0$, and thus $\mathfrak{n}_2$ is in the center of $[\mathfrak{n}, \mathfrak{n}]$. But the latter being an ideal in $\mathscr{L}$, we obtain a contradiction. Let I be a nonzero element of C, and let us write $[x, y] = B(x, y) I$ $(x, y) \in \mathfrak{n})$. B is a nondegenerate skew-symmetric bilinear form on $\mathfrak{n}/C$, and thus, if $\dim(\mathfrak{n}/C) = 2n$, we can find $2n$ elements $\{p_i, q_j; i, j = 1, 2,..., n\}$ in $\mathfrak{n}$ satisfying $[p_i, q_j] = \delta_{ij} I$. The restriction of T to exp C is not constant, hence $T \mid N$ is a factorial representation of type I (cf. [*14*]). Observe, incidentally, that what preceeds suffices to prove the Proposition if $\mathscr{L}$ is nilpotent. To settle the general case we remark, that if $\mathscr{L}$ is any solvable Lie algebra, the nilradical of which has the structure as above, G the corresponding connected and simply connected group and T a factorial representation, such that $T \mid \exp C$ is not constant, then T generates a factor of type I. In fact, to this end it is enough to observe, that the cohomology class of the extension of the irreducible representation of $T \mid N$ to G, since $G/N \sim R^m$, contains a cocycle of the form $\exp iB$, where B is a skew-symmetric bilinear form on $R^m \times R^m$, and therefore the corresponding Mackey extension is a connected nilpotent group (cf. [*2*], 188–190), and consequently T generates a factor of type I (cf. [*2*], Proposition 10.4, p. 63).

3.0. The purpose of this section is to give an outline, adapted to the needs of Section 4 below, and inspired to some extent by Part I in [*10*], of the construction, given in [*1*], of all irreducible representations of a connected and simply connected solvable group of type I.

We are going to use the following notations. Given a solvable Lie algebra $\mathscr{L}$, we write $\exp \mathscr{L}$ for the corresponding connected and simply connected group. If $\mathscr{L}_1$ is a subalgebra of $\mathscr{L}$, $\exp \mathscr{L}_1$ will stand for the connected subgroup, determined by $\mathscr{L}_1$, of $\mathscr{L}$. We recall, that since $\mathscr{L}$ is solvable, $\exp \mathscr{L}_1$ is closed and simply connected. The complexification of $\mathscr{L}$ will be denoted by $\mathscr{L}_C$. We write $\sigma(a)$ for the operator, corresponding to $a \in \exp \mathscr{L}$ in the adjoint representation, and we set $\rho(a) = (\sigma(a^{-1}))'$; ρ is the coadjoint representation of $\exp \mathscr{L}$.

Although in this paper we are concerned with solvable Lie algebras having faithful algebraic representation, the following considerations are valid for any solvable $\mathscr{L}$, such that $\exp \mathscr{L}$ is of type I.

3.1. Given any element f in the dual $\mathscr{L}'$ of $\mathscr{L}$, there exist (cf. [*1*], 2) a complex subalgebra $\mathscr{f}$ of $\mathscr{L}_C$ satisfying the following conditions.

I. $\mathscr{f}$ is maximal self orthogonal with respect to the skew-symmetric bilinear form $([x, y], f)$ on $\mathscr{L}_C$ $(x, y \in \mathscr{L}_C)$. Observe, that the radical of this form is R_C, where R is the Lie algebra of the stabilizer S of f with respect to ρ; in other words $R = \{r; r \in \mathscr{L}, (adr)'f = 0\}$. Consequently we have $\mathscr{f} \supset R_C$.

II. For all s in S, $\sigma(s)\mathscr{f} = \mathscr{f}$.

III. α) $\mathscr{f} + \bar{\mathscr{f}}$ is a subalgebra of $\mathscr{L}_C$.

β) If $x, y \in \mathscr{L}$ and $x + iy \in \mathscr{f}$, we have $([x, y], f) \geqslant 0$, and $([x, y], f) = 0$ implies, that x and y belong to $\mathscr{f} \cap \mathscr{L}$.

γ) Denoting by $\mathfrak{n}$ the nilradical (greatest nilpotent ideal) of $\mathscr{L}$, $\mathscr{f} \cap \mathfrak{n}_C$ is maximal self orthogonal in $\mathfrak{n}_C$ with respect to the restriction of the above skew symmetric bilinear form to $\mathfrak{n}_C$.

3.2. Given $\mathscr{f}$ and f as above, one can associate with them a family of irreducible unitary representations of $\exp \mathscr{L}$ (cf. 3.4. below). In order to obtain these, we need the following facts, implied more or less directly by the above conditions on $\mathscr{f}$.

a) Let us write $d = \mathscr{f} \cap \mathscr{L}$; by virtue of II this is a subalgebra, invariant with respect to the restriction of σ to S, of $\mathscr{L}$. This implies, that the subgroup $D = \exp d$ of G is normalized by S, and thus we can form the subgroup $A = DS$; one can show, that this is closed. If $\mathscr{L} = (a)$, this follows from the easily verifiable fact, that if $\mathscr{f}$ satisfies condition I in 3.1, then the image, under any faithful algebraic representation of $\mathscr{L}$, of d is algebraic.

b) Denoting by S_c the connected component of the identity in S, we have $S_c = \exp R \subset D$. One has even $S \cap D = S_c$, and therefore $A_c = D$.

c) Let us write $\hat{R} = \{r; r \in R, (r, f) = 0\}$; the derived group of S is contained in $\exp R$.

d) By virtue of I in 3.1, the linear form $l \rightarrow (l, f)$ $(l \in d)$ vanishes on the first derived algebra of d. Therefore, since D is simply connected, there exist a well determined character χ_0 on D satisfying $\chi_0(\exp l) = \exp[i(l, f)]$ $(l \in d)$. Using b) and c) above, one sees at once, that χ_0 extends to a character of A. There is a natural bijection between the set F of all such extensions of χ_0, and points of the dual of $A/A_c = S/S_c$. But since S/S_c, if not trivial, is isomorphic to the m-fold direct product of the additive group of the integers, this dual is a multitorus of dimension m.

e) If χ is an element of F, we can form the representation T_1, induced by χ, of $G = \exp \mathscr{L}$; we shall write also $T_1 = \mathrm{ind}_{A\uparrow G}\,\chi$. We recall briefly the construction of T_1 (cf. [3], pp. 80-83). Let dx and da be right invariant Haar measures on G and A respectively; we put $d(x_0x) = \Delta_G(x_0)\,dx$ and $d(a_0a) = \Delta_A(a_0)\,da$ $(x_0 \in G,\ a_0 \in A)$. Let $f(x)$ be a complex-valued measurable function on G, such that $|f(x)|^2$ is locally integrable with respect to dx, and such that for any a in A and any x in G we have

$$f(ax) = \Delta_A^{1/2}(a)\,\Delta_G^{-1/2}(a)\,\chi(a) f(x). \tag{1}$$

Writing $d\mu = |f|^2\,dx$, we have for any function h, which is continuous and of a compact support on G:

$$\int_G h(a^{-1}x)\,d\mu = \Delta_A(a)\int_G h(x)\,d\mu.$$

From this we conclude, that there exist a positive measure ν_f on the right coset space G/A of G according to A, such that

$$\int_G h(x)\,d\mu = \int_{G/A}\left(\int_A h(ax)\,da\right) d\nu_f\,.$$

We denote by $F_1(\chi, G)$ the linear space of all functions satisfying (1), and for which the total mass $\nu_f(G/A)$ of G/A with respect to ν_f is finite. Then one can define on the quotient space, according to the subspace of elements satisfying $\nu_f(G/A) = 0$, of $F_1(\chi, G)$ the structure of a Hilbert space, such that the square of the norm of the equivalence class containing f is $\nu_f(G/A)$; we denote this space by $F(\chi, G)$. The operators of the representation T_1 act on it in the following fashion: if g is in G, $T_1(g)$ is the operator, corresponding to the map $f(x) \to f(xg)$ of $F_1(\chi, G)$ into itself, on $F(\chi, G)$.

3.3. Before proceeding to complete the description of the irreducible representations, corresponding to $\not f$ and f, we make the following observation, which will be used later too (cf. Section 5). Let $\mathscr{L}_1$ be a subalgebra of $\mathscr{L}$, and α a root of $\mathscr{L}_1$ (cf. 1 above); Then there exist a root β of $\mathscr{L}$, the restriction of which to $\mathscr{L}_1$ is α. This follows from the following simple statement, the verification of which we leave to the reader. Let $\{\varphi_j; j = 1, 2,..., n\}$ and $\{\phi_k; k = 1, 2,..., n\}$ be families of complex valued linear forms on the finite dimensional real vector space V and on its subspace W respectively, such that for each w in W, the set of complex numbers $\{\phi_k(w);\ k = 1, 2,..., n\}$ is a subset of $\{\varphi_j(w); j = 1, 2,..., n\}$. Then there exist an injection r of the set $\{1, 2,..., n\}$ into $\{1, 2,..., n\}$ such that ϕ_j is the restriction of $\varphi_{r(j)}$ to $W(j = 1, 2,..., N)$.

The conclusion we wish to draw now from this is as follows: there exist a holomorphic character ψ of $G_C = \exp \mathscr{L}_C$, the restriction of which to $D \subset G \subset G_C$ coincides with $\Delta_A^{1/2}$. In fact, we obtain the restriction of $\Delta_A^{1/2}$ to D by exponentiating the linear form $l \to \frac{1}{2}\mathrm{Tr}(\mathrm{ad} l)$ $(l \in d)$. But by what we saw above, this can be viewed as the restriction to d of a linear form on $\mathscr{L}$, which vanishes on $[\mathscr{L}, \mathscr{L}]$. Extending this to $\mathscr{L}_C$ and exponentiating, we obtain a holomorphic character ψ of G_C with the desired property. From this we deduce at once, that there exist a holomorphic character ω of G_C, the restriction of which to D coincides with $\Delta_A^{1/2}\Delta_G^{-1/2}$. In fact, by what preceeds, to obtain ω it suffices to find a holomorphic character ζ of G_C, the restriction of which to G yields $\Delta_G^{1/2}$, and then set $\omega = \psi/\zeta$. But we obtain such a ζ by exponentiating the linear form $l \to \frac{1}{2}\mathrm{Tr}(\mathrm{ad}\ l)$ on $\mathscr{L}_C$.

3.4. a) We write $e = (\not{f} + \bar{\not{f}}) \cap \mathscr{L}$; by virtue of III. α) in 3.1 this is a subalgebra of $\mathscr{L}$. Let us observe now, that $\not{f} + e = e_C$. In fact, y belongs to e if and only if there exist an x in e such that $x + iy \in \not{f}$; therefore $\not{f} \subset e_C$ and $\not{f} + e \subset e_C$. Conversely, if x and y are elements of e, we can write $x + iy = (x - x_1) + (x_1 + iy)$, and the right-hand-side is in $e + \not{f}$, if x_1 is such in e, that $x_1 + iy \in \not{f}$; therefore $e_C \subset \not{f} + e$. From this we conclude, that if $H = \exp \not{f} \subset G_C$, and if O is an open set in $E = \exp e$, the set HO is open in $E_C = \exp e_C$.

b) We observe, that the linear form $l \to i(l, f)$ on $\not{f}$ vanishes on $[\not{f}, \not{f}]$ (cf. I in 3.1 above), and thus there exist a holomorphic character φ, satisfying $d\varphi(l) = i(l, f)$ $(l \in \not{f})$, on H. Let us remark, that the restriction of φ to D coincides with χ_0, as defined in 3.2. d).

c) Since $\not{f} \cap e = d$, we can find an open neighborhood of the identity U_E in E, such that $U_E^{-1} = U_E$, and that $H \cap U_H{}^2 \subset D$. Keeping U_E fixed, we write U for the open set HU_E in E_C.

d) Let us choose now an element χ in F (cf. 3.2. d)). If f is in $F_1(\chi, G)$, then for any fixed g_0 in G, the expression $\omega(h)\varphi(h)f(kg_0)$, where ω is as at the end of 3.3 and h and k are arbitrary in H and U_E respectively, depends on hk only. In fact, if $h_1 \in H$ and $k_1 \in U_E$ are such, that $h_1k_1 = hk$, then by virtue of our choice of U_E there exist an element δ in D, such that $h_1 = h\delta^{-1}$, $k_1 = \delta^{-1}k$, and therefore $\omega(h_1)\varphi(h_1)f(k_1 g_0) = \omega(\delta)\Delta_A^{-1/2}(\delta)\Delta_G^{1/2}(\delta)\varphi(\delta)\chi(\delta^{-1})\omega(h)\varphi(h)f(kg_0) =$

$\omega(h)\varphi(h)f(kg_0)$. We denote by $H_1(\chi, G)$ the collection of all elements of $F_1(\chi, G)$, for which the function $hk \to \omega(h)\varphi(h)f(kg_0)$ ($h \in H$, $k \in U_E$) is holomorphic on $U = HU_E$, for any fixed g_0 in G. One can show, that the image of $H_1(\chi, G)$ in $F(\chi, G)$ is a closed subspace, independent of the particular choice of ω and U_E; we shall denote it by $H(\chi, G)$. Since for any fixed g in G the map $f(x) \to f(xg)$ of $F_1(\chi, G)$ into itself leaves $H_1(\chi, G)$ invariant, the representation $T_1 = \mathrm{ind}_{A\uparrow G}\, \chi$ (cf. 3.2)) is reduced by $H(\chi, G)$; we denote the part of T_1 in $H(\chi, G)$ by $\mathrm{ind}(\not{f}, f, \chi)$.

3.5. Using the above notations, we can summarize the results, to be used in the sequel, of [1] in the following fashion. If $\not{f}$ is as in 3.2, then the unitary representation $\mathrm{ind}(\not{f}, f, \chi)$ is irreducible, and any irreducible unitary representation of G can be written in this form. If f_1 and f_2 in $\mathscr{L}'$ do not lie on the same orbit of ρ, then $\mathrm{ind}(\not{f}_1, f_1, \chi_1)$ and $\mathrm{ind}(\not{f}_2, f_2, \chi_2)$ are inequivalent for any choice, as above, of $\not{f}_j$ and χ_j corresponding to f_j ($j = 1, 2$). Finally, if we have $f_2 = \rho(g)f_1$ for some g in G, then making a choice, subject only to the conditions of 3.1, of $\not{f}_1$ and $\not{f}_2$, the families of representations $\mathrm{ind}(\not{f}_1, f_1, \chi_1)$ and $\mathrm{ind}(\not{f}_2, f_2, \chi_2)$, where χ_1 and χ_2 run over the collection of all admissible characters (cf. 3.2 e)), are identical.

4. From now on we shall assume, that $\mathscr{L}$ is a real solvable Lie algebra, the image of which in its adjoint representation is algebraic; we shall write for this $\mathscr{L} = (a)$ (cf. 2, 1. b)), and if $G = \exp \mathscr{L}$ is the corresponding connected and simply connected group, then $G = (a)$ (cf. 2.1. c)).

The purpose of this section is to relate an irreducible unitary representation T of G to a representation of the same kind of a subgroup of codimension 1 or 2. We are going to use this in the next section to set up an inductive procedure to determine the trace of the integral of a smooth function with respect to T (cf. 1).

We shall call a subalgebra $\not{f}$, satisfying the conditions of 3.1, of $\mathscr{L}_C$ admissible, and write $\not{f} = (A)$. Sometimes, if the context requires to emphasize, that $\not{f}$, *as a subalgebra* of $\mathscr{L}_C$, is admissible, we put $\not{f} = (A, \mathscr{L})$. If V is a real or complex finite dimensional vector space, and W a subspace of V, dim W will stand for the real or complex dimension of W according to whether V is real or complex respectively.

Let O be an orbit of ρ in $\mathscr{L}'$ (cf. the begin of 3 for the notation). We shall say, that an irreducible unitary representation T belongs to O, if T is of the form $\mathrm{ind}(\not{f}, f, \chi)$ (cf. 3.4. d)) with f in O. We recall (cf. 3.5), that by this property T is determined up to a unitary equivalence only if the stable group S of f is connected, or if O is simply connected.

4.1. With the above notations, we assume first, that there exist an ideal J, of positive dimension, in the nilradical $\mathfrak{n}$ of $\mathscr{L}$, such that J is orthogonal to f. We recall (cf. 2.2. a)), that putting $\tilde{\mathscr{L}} = \mathscr{L}/J$, we have $\tilde{\mathscr{L}} = (a)$. The dual $\tilde{\mathscr{L}}'$ of the underlying space of $\tilde{\mathscr{L}}$ can canonically be identified with the annihilator $J^{\perp}$ of J in $\mathscr{L}'$. Writing $\tilde{G} = \exp \tilde{\mathscr{L}}$, let us denote by ϕ and φ the canonical homomorphism of G and $\mathscr{L}$ onto $\tilde{G}$ and $\tilde{\mathscr{L}}$ respectively. Distinguishing notions, relative to $\tilde{G}$, by $\tilde{}$, we have $\tilde{\rho}(\phi(a)) = \rho(a) \mid J^{\perp}$ $(a \in G)$, and thus O can also be viewed as an orbit of $\tilde{\rho}$ in $\tilde{\mathscr{L}}'$.

In what follows we are going to show, that *there exist an irreducible unitary representation U, belonging to the orbit $O \subset \tilde{\mathscr{L}}'$, of G, such that* $U \circ \phi = T$. To this end, let us prove first, that if $\not{f} = (A)$ with f as above, then, denoting by φ, too, its extension to a homomorphism of $\mathscr{L}_C$ onto $\tilde{\mathscr{L}}_C$, we have $\varphi(\not{f}) = (A, \tilde{\mathscr{L}})$ with f considered as an element of $\tilde{\mathscr{L}}'$. Writing $\tilde{\not{f}} = \varphi(\not{f})$, we show, that $\tilde{\not{f}}$ satisfies all the conditions of 3.1. We observe first, that if B is a skew-symmetric bilinear form on a complex (or real) vector space, and if R is the radical of B, a self orthogonal subspace W is maximal with respect to this property if and only if we have $\dim W = \frac{1}{2}(\dim V + \dim R)$. Therefore, by virtue of I in 3.1, if R is the Lie algebra of the stable group S of f, we have $\dim \not{f} = \frac{1}{2}(\dim \mathscr{L} + \dim R)$. We have evidently $\mathscr{L} \supset R \supset J$, and $\varphi(R) = \tilde{R}$, and thus also $\dim \tilde{\not{f}} = \frac{1}{2}(\dim \tilde{\mathscr{L}} + \dim \tilde{R})$. That $\tilde{\not{f}}$ satisfies condition II follows from the relations $\phi^{-1}(\tilde{S}) = S$, and $\sigma(\phi(a)) \circ \varphi = \varphi \circ \sigma(a)$ $(a \in G)$. III α) and III β) being trivially fulfilled, we turn now to the verification of III γ). We observe first, that $\tilde{\mathfrak{n}}$ is the direct sum of $\varphi(\mathfrak{n})$ and of an ideal I of the center of $\tilde{\mathscr{L}}$. In fact, we have evidently $\varphi(\mathfrak{n}) \subset \tilde{\mathfrak{n}}$. Furthermore, since $\mathscr{L} = (a)$, $\mathscr{L} = \mathfrak{n} + \mathfrak{a}$, where $\mathfrak{a}$ is a subalgebra, such that $\mathrm{ad}(a)$ is semi-simple for any a in $\mathfrak{a}$, and $\mathfrak{n} \cap \mathfrak{a} = 0$ (cf. 2.1. b)). We have thus $\tilde{\mathscr{L}} = \varphi(\mathfrak{n}) + \varphi(\mathfrak{a})$, where the right hand side, since $J = \ker \varphi \subset \mathfrak{n}$, is a direct sum of subspaces. Therefore, writing $I = \tilde{\mathfrak{n}} \cap \varphi(\mathfrak{a})$, we obtain that $\tilde{\mathfrak{n}} = \varphi(\mathfrak{n}) + I$. Since, if $\mathrm{ad}(a)$ $(a \in \mathscr{L})$ is semi-simple, so is $\mathrm{ad}\, \varphi(a)$, for any b in I $\mathrm{ad}(b)$ is semi-simple and nilpotent, and thus $\mathrm{ad}(b) = 0$ and I is contained in the center of $\tilde{\mathscr{L}}$. We write $B(x, y) = ([x, y], f)$ $(x, y \in \mu_C)$ and $\tilde{B}(x, y) = ([x, y], f)$ $(x, y \in \tilde{\mathfrak{n}}_C)$, and denote by P and $\tilde{P}$ the radical of B and $\tilde{B}$ respectively. We have obviously $\tilde{P} = \varphi(P) + I_C$. III. γ) will be satisfied by $\tilde{\not{f}}$ if and only if $\dim(\tilde{\not{f}} \cap \tilde{\mathfrak{n}}_C) = \frac{1}{2}(\dim \tilde{\mathfrak{n}} + \dim P)$. We have $\tilde{\not{f}} \cap \varphi(\mathfrak{n}_C) = \varphi(\not{f} \cap \mathfrak{n}_C)$. In fact, if $\varphi(h) = \varphi(n) = a$

$(h \in \not{f}, n \in \mathfrak{n}_C)$, then we have $h = n + j \ (j \in J_C)$, and thus, since $J \subset \mathfrak{n}$, $h \in \not{f} \cap \mathfrak{n}_C$ and $a \in \varphi(\not{f} \cap \mathfrak{n}_C)$. Therefore

$$\begin{aligned}\dim(\tilde{\not{f}} \cap \tilde{\mathfrak{n}}_C) &= \dim(\tilde{\not{f}} \cap \varphi(\mathfrak{n}_C)) + \dim I \\ &= \dim(\not{f} \cap \mathfrak{n}_C) - \dim J + \dim I \\ &= \tfrac{1}{2}(\dim \mathfrak{n} + \dim P) - \dim J + \dim I \\ &= \tfrac{1}{2}(\dim \tilde{\mathfrak{n}} + \dim \tilde{P}),\end{aligned}$$

since obviously $J \subset P$, and thus $\dim P = \dim \varphi(P) + \dim J$.

We observe next, that since $\phi^{-1}(\tilde{S}) = S$ and $\phi^{-1}(\tilde{D}) = D$, we have $\phi^{-1}(\tilde{A}) = \phi^{-1}(\tilde{S}\tilde{D}) = SD = A$. Furthermore, there exist a character $\tilde{\chi}$ of $\tilde{A}$ with $\tilde{\chi} \circ \phi = \chi$ and $d\tilde{\chi}(l) = i(l, f) \ (l \in \tilde{d} = \tilde{\not{f}} \cap \tilde{\mathscr{L}})$. In this fashion we can form, as in 3.4. d), the unitary representation $\mathrm{ind}(\tilde{\not{f}}, f, \tilde{\chi})$ of G. This representation belongs to the orbit $O \subset \tilde{\mathscr{L}}'$. In what follows we shall show, that $\mathrm{ind}(\tilde{\not{f}}, f, \tilde{\chi}) \circ \phi = \mathrm{ind}(\not{f}, f, \chi) = T_1$ and thus $U = \mathrm{ind}(\tilde{\not{f}}, f, \tilde{\chi})$ will satisfy the conditions formulated at the beginning.

For the following reasonings cf. 2.2 in [10]. We denote again by ϕ the extension to G_C of the canonical homomorphism of G onto $\tilde{G}$. We assume to be known, that, using the notations of 3.2. e), there exist an isomorphism ψ of $F(\tilde{\chi}, \tilde{G})$ onto $F(\chi, G)$, such that $\psi T_1(\phi(a)) = T_1(a)\psi \ (a \in G)$; the image, under ψ, of the equivalence class containing $f \in F_1(\tilde{\chi}, G)$ is the equivalence class of $f \circ \phi$. (Observe, that by virtue of the definition of $F(\chi, G)$ in 3.2. e), two elements in $F_1(\chi, G)$ give rise to the same element in $F(\chi, G)$ if and only if they coincide on G almost everywhere with respect to the Haar measure). To prove our statement, it suffices to show, that $\psi H(\tilde{\chi}, \tilde{G}) = H(\chi, G)$ (cf. 3.4. d)). Writing $\eta = \Delta_A^{1/2}\Delta_G^{-1/2}$ we have $\tilde{\eta} \circ \phi = \eta$, and therefore, if $\tilde{\omega}$ is a holomorphic character of $\tilde{G}_C$, such that $\tilde{\omega} \mid \tilde{D} = \tilde{\eta}$ (cf. 3.3)), then $\omega = \tilde{\omega} \circ \phi$ is a holomorphic character, satisfying $\omega \mid D = \eta$, of G_C. We have also $\tilde{\varphi} \circ \phi = \varphi$ (cf. 3.4. b)). Let U_E be an open neighborhood of the identity of E, such that $U_E^{-1} = U_E$ and $H \cap U_E{}^2 \subset D$; then $\tilde{U}_E = \phi(U_E)$ is an open neighborhood of the identity, having analogous properties with $\tilde{H}$ and $\tilde{D}$, of $\tilde{E}$, and we have $\phi(U) = \tilde{U}$, where $U = HU_E$ and $\tilde{U} = \tilde{H}\tilde{U}_{\tilde{E}}$. Let us form now the spaces $H_1(\tilde{\chi}, \tilde{G})$ and $H_1(\chi, G)$, as in 3.4. d), and show, that $f \in H_1(\tilde{\chi}, \tilde{G})$ implies $f \circ \phi \in H_1(\chi, G)$; this will prove $\psi(H(\tilde{\chi}, \tilde{G})) \subset H(\chi, G)$. But for this it is enough to remark, that if $\tilde{K}(\tilde{u}) = \tilde{\omega}(\tilde{h})\tilde{\varphi}(\tilde{h}) f(\tilde{k}\tilde{g}_0) \ (\tilde{u} = \tilde{h}\tilde{k}, \tilde{h} \in \tilde{H}, \tilde{k} \in \tilde{U}_{\tilde{E}})$, for any fixed $\tilde{g}_0$ in $\tilde{G}$, is holomorphic on $\tilde{U}$, then the same will hold true for $K(u) = \omega(h)\varphi(h) f(\phi(kg_0)) \ (u = hk, h \in H, k \in U_E)$ on U for each fixed g_0 in G. To show, that $\psi(H(\tilde{\chi}, \tilde{G})) = H(\chi, G)$, it suffices to prove, that if $\tilde{f} \in F_1(\tilde{\chi}, \tilde{G})$, $f \in H_1(\chi, G)$, and $\tilde{f} \circ \phi = f$

almost everywhere on G, then there exist an element $\tilde{f}_1$ in $H_1(\tilde{\chi}, \tilde{G})$, such that $\tilde{f} = \tilde{f}_1$ almost everywhere on $\tilde{G}$. Since $\eta \mid \exp J \equiv 1$ and $\chi \mid \exp J \equiv 1$, we have $f(ax) = f(x)$ for all a in $\exp J$ and x in G, and thus there exist an element $\tilde{f}_1$ in $F_1(\tilde{\chi}, \tilde{G})$, such that $\tilde{f}_1 \circ \phi = f$ everywhere on G. Therefore $\tilde{f} = \tilde{f}_1$ almost everywhere on $\tilde{G}$, and to complete our proof, it is enough to show, that $\tilde{f}_1$ lies in $H_1(\tilde{\chi}, \tilde{G})$. Let us write now $K(u) = \omega(h)\varphi(h)f(kg_0)$ ($u = hk$, $h \in H$, $k \in U_E$, g_0 fix in G). We have $J_C \subset \mathscr{f}$, and $\omega \mid \exp J_C \equiv 1$, and $\varphi \mid \exp J_C \equiv 1$, and therefore $K(au) = K(u)$ for all a in $\exp J_C$ and u in U. Thus to obtain the desired conclusion, it suffices to observe, that if $\tilde{K}$ is a function on $\tilde{U}$, such that $\tilde{K} \circ \phi = K$ everywhere on U, and if K is holomorphic on U, then so is $\tilde{K}$ on $\tilde{U}$.

4.2. Let again O be an orbit, and f a fixed element in O. We assume now, that there exist no ideal, of positive dimension and orthogonal to f, but that there is a non trivial abelian ideal different from the center. In what follows, distinguishing several subcases (cf. 4.9 below for the summary), we shall prove the following statement. *Let T be an irreducible unitary representation, belonging to O, of G. Then there exist a, not necessarily connected, subgroup G_1, of codimension one or two, of G, and an irreducible unitary representation V of G, with the following properties. T is induced by V, and if $G_0 = \exp \mathscr{L}_0$ is the connected component of the unity in G_1, we have $G_0 = (a)$* (cf. 2.1. c)), *and the restriction of U of V to G_0 is irreducible. Let π be the canonical projection from $\mathscr{L}'$ onto $\mathscr{L}'_0$, and O_0 the orbit, with respect to G_0, in $\mathscr{L}'_0$, containing $\pi(f)$. Then U belongs to O_0, the complete inverse image $\pi^{-1}(O_0)$, of O_0 in $\mathscr{L}'$, is contained in O, and if O is closed, then so is O_0.*

4.3. We assume first, that there exist an ideal J, of dimension one and different from the center of $\mathscr{L}$. Let j be a non zero element in J; by virtue of our assumption, made at the begin of 4.2, we have $(j, f) \neq 0$, and thus we can assume $(j, f) = 1$. We write $\operatorname{ad} lj = \lambda(l) j$ ($l \in \mathscr{L}$), and $\mathscr{L}_0 = \ker \lambda$; observe, that $\dim \mathscr{L}_0 + 1 = \dim \mathscr{L}$.

a) We prove first, that $\operatorname{ad} \mathscr{L}_0$ (the image of $\mathscr{L}_0$ in the adjoint representation of $\mathscr{L}$) is an algebraic subalgebra of $L(\mathscr{L})$ (cf. 2.1. a)). To this end, let us observe, that since $\mathscr{L} = (a)$, $\operatorname{ad} \mathscr{L}$ is algebraic in $L(\mathscr{L})$. Furthermore $\operatorname{ad} \mathscr{L}_0 = \{a; a \in \operatorname{ad} \mathscr{L}, aj = 0\}$, which already proves our statement. From this we conclude at once, that $\mathscr{L}_0 = (a)$. In fact, one sees easily, using the characterization, given in 2.1. a), of algebraic solvable algebras, that if $\mathscr{B}$ is such an algebra, acting on a finite dimensional vector space V, and if W is a subspace, invariant under $\mathscr{B}$, of V, then $\mathscr{B} \mid W$ is algebraic in $L(W)$. To obtain $\mathscr{L}_0 = (a)$, it suffices to apply this to the case, when $V = \mathscr{L}$, $W = \mathscr{L}_0$, and $\mathscr{B} = \operatorname{ad} \mathscr{L}_0$.

b) We denote by S the stabilizer of f, and put $G_0 = \exp \mathscr{L}_0$. Then we have $S \subset G_0$. In fact, there exist an homomorphism Δ of G into the multiplicative group of positive numbers such that $\sigma(a) j = \Delta(a) j$ for all a in G, and $G_0 = \ker \Delta$. If s belongs to S, then we have $(j, f) = (j, \rho(s) f) = (\sigma(s^{-1}) j, f) = \Delta(s^{-1})(j, f)$, whence $\Delta(s) = 1$, since $(j, f) \neq 0$. Let S^0 be the stabilizer of $\pi(f)$ in G_0; by virtue of what we have just seen S^0 contains S. If $S_c{}^0$ is the connected component of the identity in S^0, then $S^0 = S \,.\, S_c{}^0$. In fact, let λ be the element, determined by $(l, \lambda) = \lambda(l)$ $(l \in \mathscr{L})$, of $\mathscr{L}'$. We have $\rho(a)\lambda = \lambda$ for all a in G. If s is in S^0, then $\rho(s) f = f + A(s)\lambda$, and $s \to A(s)$ is a homomorphism if S^0 in the additive group of the reals, such that $\ker A = S$. Putting $\tilde{r} = j$, we get $-(\operatorname{ad} \tilde{r})' f = \lambda$ and thus $\{\exp(tr);\ t \in R\} \subset S_c{}^0$ and any s in S^0 can be written as $s_0 \exp(t_0 r)$, where $s_0 \in S$, proving, that $S^0 = S . S_c{}^0$.

c) Let $\not{h}$ be a subalgebra of $(\mathscr{L}_0)_C$, such that $\not{h} = (A, \mathscr{L}_0)$ with respect to $\pi(f)$. We are going to prove, that $\not{h} = (A, \mathscr{L})$ with respect to f. To this end we show, that $\not{h}$ satisfies the conditions of 3.1. As regards I, loc. cit., it is enough to observe, that putting $S^0 = \exp R^0$ we have $R \subset R^0$, and thus $\dim R^0/R = 1$, and therefore $\dim \not{h} = \frac{1}{2}(\dim \mathscr{L}_0 + \dim R^0) = \frac{1}{2}(\dim \mathscr{L} + \dim R)$. We have evidently $\sigma(s)(\not{h}) \subset \not{h}$ for all s in S, since $S \subset S^0$. Conditions III α) and III β) are trivially fulfilled. Let $\mathfrak{n}$ and $\mathfrak{n}_0$ be the nilradical of $\mathscr{L}$ and $\mathscr{L}_0$ respectively. Since $\mathfrak{n} \subset \mathscr{L}_0$ and since $\mathfrak{n}$ is a nilpotent ideal in $\mathscr{L}$, we have $\mathfrak{n} \subseteq \mathfrak{n}_0$. But the nilradical being a characteristic ideal, $\mathfrak{n}_0$ is a nilpotent ideal in $\mathscr{L}$, and therefore $\mathfrak{n}_0 \subseteq \mathfrak{n}$, and thus $\mathfrak{n} = \mathfrak{n}_0$, implying, that III γ), too, is satisfied.

d) Since T belongs to O, and since $\not{h} = (A)$, T can be written as $\operatorname{ind}(\not{h}, f, \chi)$, where χ is a suitably chosen unitary character of $A = DS$ (cf. 3.2. d)). We have $d = \not{h} \cap \mathscr{L} = \not{h} \cap \mathscr{L}_0$, and thus $S_c{}^0 \subset D$. Therefore $A^0 = DS^0 = DS_c{}^0 S = DS = A$. Writing $f_0 = \pi(f)$, we form the representation $U = \operatorname{ind}(\not{h}, f_0, \chi)$ of G_0; it belongs to the orbit $O_0 \subset \mathscr{L}_0'$, with respect to G_0, of f_0.

e) Next we prove, following closely the reasonings of 2.2 in [*10*], that $\operatorname{ind}_{G_0 \uparrow G} U = T$. To this end, let us set $T_1 = \operatorname{ind}_{A \uparrow G} \chi$, and $U_1 = \operatorname{ind}_{A \uparrow G_0} \chi$. We recall (cf. 3.2. e)), that the representation spaces of T_1 and U_1 are given by $F(\chi, G)$ and $F(\chi, G_0)$ respectively, and that the representations T and U are obtained by restricting T_1 and U_1 to the subspaces $H(\chi, G)$ and $H(\chi, G_0)$ respectively (cf. 3.4. d)). Let us write $V = \operatorname{ind}_{G_0 \uparrow G} U$; the representation V can be realized as follows (cf. [3] p. 80). We consider first the collection of all measurable functions, with values in $F(\chi, G_0)$, on G, which satisfy $f(ax) = \Delta_A^{1/2}(a)\Delta_G^{-1/2}(a)U(a)f(x)$ for all a in A and x in G, and for which $|f|^2$ is locally summable with respect to the right invariant Haar measure dx on G (here $|\ |$ stands for the norm in $F(\chi, G_0)$). Proceeding as in 3.2. e), one can associate with the measure $|f|^2\, dx$ on G a positive measure ν_f on G/A. We denote by $H_1(U_1)$ the collection of all those functions satisfying the above conditions, for which the total mass, with respect to ν_f, of G/A is finite. Defining the norm of an element f in $H_1(U_1)$ as the positive square root of $\nu_f(G/A)$, the representation space $H(U_1)$ of V is the quotient space of $H_1(U_1)$ according to the subspace of elements having the norm zero. Finally, for any g in G the corresponding operator $V(g)$ is obtained through passage to quotient from the translation on the right by g of elements of $H_1(U_1)$.

It is well known, that the representations T_1 and V, of G, are unitarily equivalent. More precisely, reasoning as in 2.2 of [10], one can describe a unitary map W, transforming T_1 into V, of $F(\chi, G)$ onto $H(U_1)$ as follows. Given an element f in $F_1(\chi, G)$ (cf. 3.2. e)), there exist a subset $E(f)$, of measure zero, of G, such that if g does not belong to $E(f)$, the function $\phi_1(f)(g)$, defined on G_0 by $g_0 \to \Delta_{G_0}^{-1/2}(g_0)\, \Delta_G^{1/2}(g_0) f(g_0 g)$, belongs to $F_1(\chi, G)$. Denoting by $\phi(f)(g)$ its image in $F(\chi, G_0)$, and defining $\phi(f)(g)$, if $g \in E(f)$, as the zero element in $F(\chi, G_0)$, the function $g \to \phi(f)(g)$ on G belongs to $H_1(U_1)$. Finally, the image of the class, in $F(\chi, G)$, of f under W is the class of $\{\phi(f)(g)\}$ in $H(U)$.

Let us write $\tilde{T} = \operatorname{ind}_{G_0 \uparrow G} U$. $\tilde{T}$ is a subrepresentation of V, obtained by restricting it to the image $H(U)$, in $H(U_1)$, of the subspace, consisting of all elements, taking their values in $G(\chi, G_0)$ almost everywhere, of $H_1(U_1)$. To prove, that T and $\tilde{T}$ are unitarily equivalent, it is enough to show, that W maps $H(\chi, G)$ onto $H(U)$.

Let us prove first, that $WH(\chi, G) \subset H(U)$. To this end, we denote by ψ a holomorphic character of G_C, such that $\psi \mid D \equiv \Delta_D^{1/2}$, and by ζ and ζ_0 holomorphic characters of G_C and $(G_0)_C$ satisfying $\zeta \mid G \equiv \Delta_G^{1/2}$ and $\zeta_0 \mid G_0 \equiv \Delta_{G_0}^{1/2}$ respectively (cf. 3.3). Let us put $\omega(h) = \psi(h)/\zeta(h)$ and $\omega_0(h) = \psi(h)/\zeta_0(h)$ $(h \in H = \exp \not{f} \subset (G_0)_C)$. Using the notations of 3.4. d), by virtue of the definition of W it suffices to show, that if f is an element of $H_1(\chi, G)$, then for each fixed element g of G, the map $hk \to \omega_0(h)\, \varphi(h)\, \Delta_{G_0}^{-1/2}(k)\, \Delta_G^{1/2}(k) f(kg)$ $(h \in H,\ k \in U_E \subset E \subset G_0)$ is holomorphic on HU_E. But since we have

$$(\omega_0(h)/\omega(h))\, \Delta_{G_0}^{-1/2}(k)\, \Delta_G^{1/2}(k) = \zeta(hk)/\zeta_0(hk),$$

the above expression can be rewritten as $(\zeta(hk)/\zeta_0(hk))(\omega(h)\varphi(h)f(kg))$. From this our statement follows at once, since the second factor, f being in $H(\chi, G)$, is holomorphic on HU_E.

By virtue of what preceeds, to complete our proof it suffices to establish, that the image of $H(\chi, G)$ under W is dense in $H(U)$. To this end we are going to copy the argument, using Lemma 3.3 in [*12*] (p. 108), of 2.2 in [*10*], and observe, that it is enough to show, that there exist a sequence $\{f_n;\ n = 1, 2, ...,\}$ of elements in $G(\chi, G)$, and a set E of positive Haar measure in G, such that for each x in E the sequence $\{\phi(f_n)(x);\ n = 1, 2, ...,\}$ is dense in $H(\chi, G_0)$. Let $\{h_n;\ n = 1, 2, ...,\}$ be a sequence of elements in $H_1(\chi, G_0)$, such that the set of their images is dense in $H(\chi, G_0)$. Let S be a Borel section of G with respect to G_0, and S_0 a Borel subset, of compact closure, of S such that G_0S_0 be of positive Haar measure in G. For each $n = 1, 2, ...$ we define $F_n(g)$ to be $(\Delta_{G_0}^{-1/2}\Delta_G^{1/2})(g_0)h_n(g_0)$, if $g = g_0s$ with s in S_0, and set$F_n(g) = 0$ otherwise. It is easy to show, that $F_n(g)$ belongs to $F_1(\chi, G)$; but it lies even in $H_1(\chi, G)$. To this end we have to prove, that, with the notations used above, for each fixed $\bar{g}$ in G, the function $hk \to \omega(h)\,\varphi(h)\,F_n(k\bar{g})$ $(h \in H,\ k \in U_E)$ is holomorphic on HU_E. If $\bar{g} = g_0s$ $(g_0 \in G_0,\ s \in S_0)$, using a computation considered above, we obtain $\omega(h)\,\varphi(h)\,F_n(k\bar{g}) = [\zeta_0(hk)/\zeta(hk)]\,\omega_0(h)\,\varphi(h)\,f_n(kg_0)$, which, since f_n lies in $G_1(\chi, G_0)$, suffices to establish our statement. If, on the other hand, $\bar{g} = g_0s$, where s is in $S - S_0$, then $\omega(h)\,\varphi(h)\,F_n(k\bar{g}) = 0$ $(h \in H,\ k \in U_E)$. We have also, if $g = \bar{g}_0s$, with $\bar{g}_0$ in G_0 and s in S_0, that the value of the function $\phi_1(F_n)(g)$ at $g_0 \in G_0$ equals

$$(\Delta_{G_0}^{-1/2}\Delta_G^{1/2})(g_0)\,F_n(g_0g) = f_n(g_0\bar{g}_0),$$

and $\phi(F_n)(g) = 0$ otherwise. Therefore, for each g in G_0S_0, the sequence $\{\phi(F_n)(g);\ n = 1, 2, ...,\}$ is dense in $H(\chi, G_0)$, by virtue of our choice of the sequence $\{f_n;\ n = 1, 2, ...,\}$ in $H_1(\chi, G_0)$. Summing up, we have, that $WH(\chi, G) = H(U)$, implying, that $T = \mathrm{ind}_{G_0\uparrow G}\, U$.

f) Let us show next, that the complete inverse image $\pi^{-1}(O_0)$, of O_0 in $\mathscr{L}'$, is contained in O. To this end it is enough to show, that $\pi^{-1}(O_0) = \{l';\ l' = \rho(g_0)f, g_0 \in G_0\}$. Using the notations of b) above, we have an element $\bar{r}$ in $\mathscr{L}_0$, such that $\rho(\exp(t\bar{r}))f = f + t\lambda$ $(t \in R)$. If l' is in $\pi^{-1}(O_0)$, it can be written as $\rho(g_0)f + a\lambda$ $(g_0 \in G_0,\ a \in R)$, and therefore $l' = \rho(g_0)(f + a\lambda) = \rho(g_0')f$ with $g_0' = g_0\exp(a\bar{r})$ in G_0, proving our statement.

Finally, let us assume, that O is closed; we are going to prove, that this implies, that O_0, too, is closed. To this end, after what we saw above, it is enough to show, that any G_0 orbit O_1, contained in O, and having the codimension 1, is closed. If O_1 is not closed, its boundary ∂O_1, which is invariant with respect to G_0, is non empty and is contained in O. Since O cannot contain any G_0 orbit, the codimension of which (in O_1) exceeds the codimension of G_0 in G, to achieve our goal it suffices to prove, that the codimension of any G_0 orbit, contained in ∂O_1, is smaller, than $\dim O_1$.

Before proceeding, let us recall the theorem of Tarski–Seidenberg (cf. Theorem 3 in [18]). Given a finite system Σ of relations of the form $\mathscr{P}(x, y) \geqslant 0$, where $\mathscr{P}(x, y)$ is a real polynomial in the $M + N$ real variables $x = (x_1, x_2, \ldots, x_M)$, $y = (y_1, y_2, \ldots, y_N)$, there exist a subset U of R^M, which is union of a finite number of sets of the form $\{x; \mathscr{P}(x) = 0, Q_j(x) > 0 \text{ for } j = 1, 2, \ldots, J\}$, where $\mathscr{P}$ and Q_j are polynomials, such that, for a given x, Σ has solutions in y, if and only if x belongs to U.

We recall, that $G_0 = \exp \mathscr{L}_0$, and that ad $\mathscr{L}_0$ is algebraic in $L(\mathscr{L})$, (cf. a)). Therefore $(\operatorname{ad} \mathscr{L}_0)' = \{a;\ a = (\operatorname{ad} l)',\ l \in \mathscr{L}_0\}$ is algebraic in $L(\mathscr{L}')$, and thus there exist an algebraic subgroup $\mathfrak{G}$ of $GL(\mathscr{L}')$, such that the connected component of $\mathfrak{G}$ is $\rho(G_0)$ (cf. 2.1. a)). Let us write $n = \dim \mathscr{L}$, and let us select a base in $\mathscr{L}'$. We assume, that $\mathfrak{G}$ is the collection of all those elements in $GL(\mathscr{L}')$, the matrix coefficients of which, taken with respect to the said base in $\mathscr{L}'$, satisfy $\mathscr{P}_k(a) = 0$ $(k = 1, 2, \ldots, K)$, where $\mathscr{P}_k$'s are polynomials. Let the components of f (f fixed in O) be $\{x_k{}^0; k = 1, 2, \ldots, n\}$, and let us consider the following system of polynomial relations in the $n + n^2$ real variables $\{x_k, a_{ij}; i, j, k = 1, 2, \ldots, n\}$: $0 = x_i - \sum_{j=1}^{n} a_{ij} x_j{}^0$ $(i = 1, 2, \ldots, n)$, $\mathscr{P}_k(a) = 0$ $(k = 1, 2, \ldots, K)$ and $\det a \neq 0$ $(a = (a_{ij}))$. Denoting the orbit, containing O_1, of $\mathfrak{G}$, by $\tilde{O}$ and using the theorem of Tarski–Seidenberg, we conclude, that $\tilde{O}$ is a finite union of sets of the form $\{x; \mathscr{P}(x) = 0, Q_j(x) > 0 \text{ for } j = 1, 2, \ldots, J\}$. But since O_1 is one of the finite number of connected components of $\tilde{O}$, we see easily, that any G_0 orbit O_2 contained in the boundary of O_1 satisfies $\dim O_2 < \dim O$.

Remark. The argument just presented proves, that if $G_0 = \exp \mathscr{L}_0$ is a subgroup of arbitrary codimension, such that ad $\mathscr{L}_0$ is algebraic in $L(\mathscr{L})$, then any G_0 orbit O_0, satisfying $\operatorname{codim} O_0 = \operatorname{codim} G_0$, of O is closed.

Summing up, under the assumption made at the begin of 4.3 we can satisfy the conditions of 4.2 by choosing $G = G_0$, and $U = V$ (cf. d)). In fact, U belongs to O_0, we have $\operatorname{ind}_{G_0 \uparrow G} U = T$ (cf. e)), $\pi^{-1}(O_0)$ is contained in O, and if O is closed, then so is O_0 (cf. f)).

4.4. Let us assume next, that there exist no ideal, of dimension one and different from the center, in $\mathscr{L}$, but that there exist an ideal J

of dimension two, containing a base $\{j_1, j_2\}$, such that $\mathrm{ad} l j_1 = \alpha(l) j_2$, $\mathrm{ad} l j_2 = -\alpha(l) j_1$ $(l \in \mathscr{L})$. Writing $\mathscr{L}_0 = \ker \alpha$, we have $\dim \mathscr{L}_0 + 1 = \dim \mathscr{L}$.

a) Observing, that $\mathscr{L}_0$ is the centralizer of j_1(say), we conclude, that $\mathrm{ad}\, \mathscr{L}_0$ is algebraic in $L(\mathscr{L})$, and hence we show, as in 4.3. a), that $\mathscr{L}_0 = (a)$.

b) Let us assume now, that $\not{f} = (A, \mathscr{L})$ (cf. the begin of 4), and let us prove, that this implies, that $\not{f}$ is contained in $(\mathscr{L}_0)_C$. To this end let us observe first, that $J_C \subset \not{f}$ implies $\not{f} \subset (\mathscr{L}_0)_C$. In fact, we have in this case $0 = ([h, j_1], f) = \alpha(h)(j_2, f)$ and $0 = ([h, j_2], f) = -\alpha(h)(j_1, f)$ for all h in $\not{f}$. But by virtue of the assumption, made at the begin of 4.2, J is not orthogonal to f, and thus $\not{f} \subset \ker \alpha$. We denote by $\mathfrak{n}$ the nilradical of $\mathscr{L}$ and recall, that in consequence of condition III. γ) in 3.1, $\not{f} \cap \mathfrak{n}_C$ is maximal self orthogonal with respect to the skew symmetric form $([x, y], f)$ on $\mathfrak{n}_C \times \mathfrak{n}_C$. Let us assume now, that $\not{f} \not\subset (\mathscr{L}_0)_C$; then also $J_C \not\subset \not{f}$. Let us write $\not{f}_1 = (\not{f} \cap (\mathscr{L}_0)_C) + J_C$; one sees at once, that $\not{f}_1 \cap \mathfrak{n}_C$ is self orthogonal. But since $J \subset \mathfrak{n}$, and $\mathfrak{n} \subset \mathscr{L}_0$, we have $\not{f}_1 \cap \mathfrak{n}_C = \not{f} \cap \mathfrak{n}_C + J_C$ and thus $\dim(\not{f}_1 \cap \mathfrak{n}_C) > \dim(\not{f} \cap \mathfrak{n}_C)$, giving a contradiction.

c) We write $G_0 = \exp \mathscr{L}_0$, and assume, that S, S^0, R and R^0 have a meaning analogous to that in 4.3. b). We show, that $S^0 = (S \cap G_0).S_c^{\,0}$. In fact, let us denote by α the element, determined by $(l, \alpha) = \alpha(l)$ $(l \in \mathscr{L})$, of $\mathscr{L}_0^{\perp} \subset \mathscr{L}'$. Then, for any s in S^0 we have $\rho(s) f = f + A(s)\alpha$, where $A(s)$ is a homomorphism of S^0 into the additive group of the reals, the kernel of which is $G_0 \cap S$. Observe that, since J is not orthogonal to f, by replacing, if necessary, j_1 and j_2 by $aj_1 + bj_2$ and $-bj_1 + aj_2$ respectively, through an appropriate choice of the numbers a and b we can always achieve, that $(j_1, f) = 1$, $(j_2, f) = 0$. Assuming this, and putting $\bar{r} = j_1$, we get $-(\mathrm{ad}\, \bar{r})' f = \alpha$, and consequently $A(\exp(t\bar{r})) \equiv t$ $(t \in R)$ which, since $S_c^{\,0} = \exp R^0$ already implies the desired conclusion.

d) We know (cf. b), that $\not{f} = (A, \mathscr{L})$ implies $\not{f} \subset (\mathscr{L}_0)_C$; let us prove, that we have also $\not{f} = (A, \mathscr{L}_0)$. In fact, conditions I, III α) and III β) in 3.1 are trivially fulfilled. In particular (by virtue of I) we have $R^0 \subset \not{f}$, and thus $S_c^{\,0}$ normalizes $\not{f}$. But then, since $S^0 = (S \cap G_0) S_c^{\,0}$ (cf. c)), so does S^0; hence condition II loc. cit., too, is satisfied. Finally, reasoning, as in 4.3. c) we show, that the nilradicals of $\mathscr{L}$ and $\mathscr{L}_0$ coincide, implying condition III γ).

e) We write again $d = \not{f} \cap \mathscr{L} = \not{f} \cap \mathscr{L}_0$, $D = \exp d$, $A = SD$ and $A^0 = S^0 D$. Observe, that $S^0 D = (S \cap G_0) S_c^{\,0} D = (S \cap G_0) D$, whence we conclude easily, that $A^0 = A \cap G_0$.

Since T belongs to O, there exists a unitary character χ of A such that $T = \operatorname{ind}(\not{f}, f, \chi)$ (cf. 3.4. d). Writing $\chi^0 = \chi \mid A^0$, since $\not{f} = (A, \mathscr{L}_0)$ and $A_c^0 = A_c$, we can form the irreducible representation $U = \operatorname{ind}(\not{f}, f_0, \chi^0)$ of G_0 ($f_0 = \pi(f)$); it belongs to O_0.

f) We denote by A the homomorphism, satisfying $dA = \alpha$, of G into the additive group of the reals; we have $\ker A = G_0$. Let $\{j_1', j_2'\}$ be a base, dual to $\{j_1, j_2\}$, in the dual $J' = \mathscr{L}'/J^\perp$ of J. If $a \in G$, the matrix, with respect to this base, of the operator $\rho(a) \mid J'$ has the form

$$\begin{pmatrix} \cos A(a) & -\sin A(a) \\ \sin A(a) & \cos A(a) \end{pmatrix}.$$

Since the projection of f onto J' is nonzero, we conclude, that, for any s in S, we have $\rho(s) \mid J' =$ unity, or $A(s) = 2\pi k$, where k is some integer. We deduce from all this, that $G_1 = SG_0$ is a closed invariant subgroup of G.

g) Let us form the unitary representations $V_1 = \operatorname{ind}_{A \uparrow G_1} \chi$ and $U_1 = \operatorname{ind}_{A^0 \uparrow G_0} \chi^0$, acting on the Hilbert spaces $F(\chi, G)$ and $F(\chi^0, G_0)$ respectively (cf. 3.2. e)). We are going to show, that there exist a unitary map W from $F(\chi, G_1)$ onto $F(\chi^0, G_0)$, such that W carries the restriction of V_1 to G_0 into U_1. To this end, given an element f in $F_1(\chi, G)$, let us write $W'f$ for the restriction of f to G_0. We show first, that $WF_1(\chi, G) = F_1(\chi^0, G_0)$, and that W' preserves the norm. Since G_1/G_0 is discrete, it is clear, that $f' = W'f$ is measurable, and that $|f'|^2$ is locally integrable on G_0 with respect to a right invariant Haar measure. Furthermore, since $\chi \mid G_0 \equiv \chi^0$, $\Delta_A \mid A^0 \equiv \Delta_A$ and $\Delta_{G_1} \mid G_0 \equiv \Delta_{G_0}$, we have $f'(ax) = \Delta_{A^0}^{1/2}(a)\, \Delta_{G_0}^{-1/2}(a)\, \chi^0(a) f'(x)$ for any a in A^0 and x in G_0. Let us form now, as in 3.2. e) the measure ν_f on G_1/A. It is easy to see, that its image ν, under the canonical bijection between G_1/A and G_0/A^0, satisfies

$$\int_{G_0} h(x)|f'(x)|^2 \, dx = \int_{G/A^0} \left(\int_{A^0} h(ax) \, da \right) d\nu$$

where h is continuous with compact support on G_0, provided the invariant measures dx and da have been appropriately normalized. Using these measures to form $F_1(\chi^0, G_0)$, we can conclude, that f' lies in $F_1(\chi^0, G_0)$, and that its norm coincides with that of f in $F_1(\chi, G)$. Let us show finally, that given f' in $F_1(\chi^0, G_0)$, there exist an f in $F_1(\chi, G)$ such that $W'f = f'$. To this end, let us write $\psi(a) = \Delta_A^{1/2}(a)\, \Delta_{G_0}^{-1/2}(a)\, \chi(a)$ $(a \in A)$. One sees easily, using $A^0 = A \cap G_0$ (cf. e)), that if $ag_0 = a'g_0'$ $(a, a' \in A, g_0, g_0' \in G_0)$ then also $\psi(a) f'(g_0) =$

$\psi(a')f'(g_0')$, and thus there exist a well determined function f on $AG_0 = DSG_0 = G_1$, such that $f(ag) = \psi(a)f(g)$ and $f \mid G_0 \equiv f'$. It is immediate, that $f \in F_1(\chi, G_1)$ and $W'f = f'$. We denote by W the unitary map, which assigns to the class of $f \in F_1(\chi, G)$ in $F(\chi, G)$ the class of $W'f$ in $F(\chi^0, G_0)$, from $F(\chi, G_1)$ onto $F(\chi^0, G_0)$. Since evidently W' commutes with translations on the right by elements of G_0, W carries the restriction of V to G_0 into U.

h) Let U_E be as in 3.4. c), ω and φ as in 3.4. d), and let us denote by $H_1(\chi, G_1)$ the collection of all those elements in $F_1(\chi, G_1)$, for which the map $hk \to \omega(h)\,\varphi(h)\,f(hkg_1)$ $(h \in H = \exp \mathscr{h},\ k \in U_E)$ for each fixed g_1 in G_1 is holomorphic on HU_E. We denote by $H(\chi, G_1)$ the closure of the image of $H_1(\chi, G_1)$ in $F(\chi, G_1)$. This is a closed subspace, invariant by V_1; let us write V for the part of V_1 in $H(\chi, G_1)$. An easy modification of the argument of 4.3. e) shows, that $\mathrm{ind}_{G_1 \uparrow G} V$ is unitarily equivalent to $T = \mathrm{ind}(\mathscr{h}, f, \chi)$. On the other hand, W' (cf. g)) maps $H_1(\chi, G_1)$ into $H_1(\chi^0, G_0)$, and therefore W maps $H(\chi, G_1)$ into $H(\chi^0, G_0)$. But since $H(\chi, G_1)$ is of a positive dimension, $WH(\chi, G_1)$ is invariant by U_1 and $U_1 \mid H(\chi^0, G_0) = U$ (cf. e)) is irreducible, we conclude, that $WH(\chi, G_1) = H(\chi^0, G_0)$, and that the restriction of V to G_0 and U are unitarily equivalent.

Summing up, we have found an irreducible unitary representation V of G_1 such that, up to unitary equivalence, we have $\mathrm{ind}_{G_1 \uparrow G} V = T$, and that the restriction U of V to G_0 is irreducible and belongs to the orbit $O_0 \subset \mathscr{L}_0'$.

i) The proof, that $\pi^{-1}(O_0)$ is contained in O, and that O_0 is closed, if so is O, is very much the same as in 4.3. f). We show first, using a reasoning of c) above, that $\pi^{-1}(O_0) = O_1$ is the G_0 orbit, containing f, in $\mathscr{L}'$. If O is closed, then by virtue of a) and the Remark at the end of 3.3. f) O_1, too, is closed, and thus so is O_0.

4.5. Next we assume, that $\mathscr{L}$ does not contain ideals like those considered in 4.3 and 4.4, but that there exist an ideal J, of dimension two, containing a base $\{j_1, j_2\}$ such that $\mathrm{ad}l j_1 = \alpha_1(l)\,j_1 + \alpha_2(l)\,j_2$, $\mathrm{ad}l j_2 = -\alpha_2(l)\,j_1 + \alpha_1(l)\,j_2$ $(l \in \mathscr{L})$ and that $\alpha_1 \not\equiv 0$ and $\alpha_2 \not\equiv 0$.

a) Let us show first, that the linear forms α_1 and α_2 on $\mathscr{L}$ are linearly independent. In fact, since ad $\mathscr{L}$ is algebraic in $L(\mathscr{L})$, ad $\mathscr{L} \mid J$ is abelian algebraic in $L(J)$ (cf. 4.3. a)) consisting of semi-simple endomorphisms. Therefore (cf. 2.1. a)) there exist an element $l = l_1 + il_2$ $(l_k \in \mathscr{L},\ k = 1, 2)$ in $\mathscr{L}_C$, such that $\alpha_1(l) + i\alpha_2(l) = n_+$ and $\alpha_1(l) - i\alpha_2(l) = n_-$ are integers, and n_+ and n_- are not zero simultaneously. If $\alpha_1 = c\alpha_2$ $(c \neq 0)$, this implies, that $\alpha_2(l)(c \pm i) = n_\pm$

and thus $(c + i)/(c - i)$ is real, which is impossible, since c is real. We write $\mathscr{L}_0 = \ker \alpha_1 \cap \ker \alpha_2$; according to what we have just seen, $\mathscr{L}_0$ is an ideal, of codimension two, in $\mathscr{L}$. Since $\mathscr{L}_0$ is the centralizer of j_1, say, we conclude as in 4.3. a), that ad $\mathscr{L}_0$ is algebraic in $L(\mathscr{L})$, and that $\mathscr{L}_0 = (a)$.

b) We assume, that $\not{f}$ is a subalgebra of $\mathscr{L}_C$, such that $\not{f} = (A)$, and we show, that this implies $\not{f} \subset (\mathscr{L}_0)_C$. Let us observe, that if $J_C \subset \not{f}$, then $\not{f} \subset (\mathscr{L}_0)_C$. In fact, under the said assumption we have for all h in $\not{f}$

$$0 = ([h, j_1], f) = \alpha_1(h)(j_1, f) + \alpha_2(h)(j_2, f),$$
$$0 = ([h, j_2], f) = -\alpha_1(h)(j_2, f) + \alpha_2(h)(j_1, f).$$

But since J is not orthogonal to f, we have $(j_1, f)^2 + (j_2, f)^2 \neq 0$, and thus $\alpha_1(h) = \alpha_2(h) = 0$ and $h \in (\mathscr{L}_0)_C$. From here we can finish proving, that $f \subset (\mathscr{L}_0)_C$ as in 4.4. a).

c) Let us write again $G_0 = \exp \mathscr{L}_0$, and let us assume, that S, S^0, R and R^0 are defined similarly as in 4.3. b). We are going to prove, that $S^0 = (S \cap G_0) S_c^0$. We denote by α_k the elements, determined by $(l, \alpha_k) = \alpha_k(l)$ $(l \in \mathscr{L};\ k = 1, 2)$ of $\mathscr{L}'$; obviously, α_1 and α_2 form a base in $\mathscr{L}_0^{\perp}$. We write, for any s in S^0: $\rho(s) f = f + A_1(s)\,\alpha_1 + A_2(s)\,\alpha_2$. Since the restriction of $\rho(s)$ to $\mathscr{L}_0^{\perp}$ is the identity, we conclude, that $A_k(s)$ is a homomorphism of S^0 into the additive group of the reals $(k = 1, 2)$, and that $\ker A_1 \cap \ker A_2 = S \cap G_0$. We can assume, as in 4.4. c), that $(j_1, f) = 1$ and $(j_2, f) = 0$. Putting $j_1 = \bar{r}_1$ and $-j_2 = \bar{r}_2$ we get $-(\mathrm{ad}\, \bar{r}_k)' f = \alpha_k$ $(k = 1, 2)$ and consequently $\rho(\exp(t_1 \bar{r}_1) \exp(t_2 \bar{r}_2)) f = f + t_1 \alpha_1 + t_2 \alpha_2$ $(t_1, t_2 \in R)$. From here we can complete the proof, that $S^0 = (S \cap G_0)\, S_c^0$ as in 4.3. b).

d) If $\not{f} = (A, \mathscr{L})$, then by b) above $\not{f} \subset (\mathscr{L}_0)_C$. Using c) and the reasonings of 4.4. d) one shows easily, that even $\not{f} = (A, \mathscr{L}_0)$.

e) We write $A = SD$, $A^0 = S^0 D$, and observe, that by virtue of $S^0 = (S \cap G_0) S_c^0$ we have $A \cap G_0 = A^0$. Assuming $T = \mathrm{ind}(\not{f}, f, \chi)$, putting $\chi^0 = \chi \mid A^0$ and $f_0 = \pi(f) \in \mathscr{L}_0'$, we form the irreducible unitary representation $U = \mathrm{ind}(\not{f}, f_0, \chi^0)$ of G_0; it belongs to O_0.

f) If $\{j_1', j_2'\}$ is a base, dual to $\{j_1, j_2\}$, in $J' = \mathscr{L}'/J^{\perp}$, for any a in G the matrix expression, with respect to this base, of $\rho(a) \mid J'$ has the form

$$\begin{pmatrix} \cos A_2(a) & -\sin A_2(a) \\ \sin A_2(a) & \cos A_2(a) \end{pmatrix} e^{-A_1(a)}.$$

Here A_k is a homomorphism, satisfying $dA_k = \alpha_k$, of G into the additive group of the reals ($k = 1, 2$). Since the projection of f onto J' is nonzero, for any s in S we have $A_1(s) = 0$ and $A_2(s) = 2\pi k$, where k is some integer. We have in addition $G_0 = \ker A_1 \cap \ker A_2$, from which we conclude, that $G_1 = SG_0$ is a closed invariant subgroup of G, such that G_1/G_0 is discrete.

g) Proceeding precisely as in 4.4. g) and 4.4. h), we can define a unitary representation V of G, such that $\mathrm{ind}_{G_1\uparrow G} V = T$, and the restriction to G_0 is unitarily equivalent to U.

h) We can show as in 4.3. f), using c) above, that $\pi^{-1}(O_0) = O_1$ is the G_0 orbit, containing f, in $\mathscr{L}'$. The codimension of O_1 in O and of G_0 in G both equal two, and therefore we can use the Remark in 4.3. f), taking into account a), to show, that if O is closed, then so is O_1 and O_0.

4.6. In 4.3–4.5 we assumed, that there existed minimal ideals not contained in the center of $\mathscr{L}$. Next we consider cases, when such ideals do not exist. Then, in particular, the center is nontrivial and we observe, that by virtue of the assumption, made at the beginning of 4.2, according to which there exist no ideals, orthogonal to f, in $\mathscr{L}$, its dimension equals one.

Let us assume now, that there is an ideal J, of dimension two, of $\mathscr{L}$, and let $\{c, g\}$ be a base in J, such that c is in the center. Then we have $(c, f) \neq 0$, and can obviously assume, that $(g, f) = 0$ and $(c, f) \neq 1$. For any l in $\mathscr{L}$ we write $\mathrm{ad} l g = \delta(l)\, g + \gamma(l)\, c$; by virtue of the assumption made above, we have $\gamma \not\equiv 0$.

a) We write $\mathscr{L}_0 = \ker \gamma$; $\mathscr{L}_0$ is a subalgebra of $\mathscr{L}$, and $\dim \mathscr{L}_0 + 1 = \dim \mathscr{L}$. We observe, that $\mathrm{ad}\, \mathscr{L}_0$ is an algebraic subalgebra in $L(\mathscr{L})$. In fact, denoting by Γ the one dimensional subalgebra spanned by g, we have $\mathrm{ad}\, \mathscr{L}_0 = \{a;\ a \in \mathrm{ad}\, \mathscr{L},\ a\Gamma \subseteq \Gamma\}$. If l_0 is in $\mathscr{L}_0$, and b is a replica of $\mathrm{ad}\, l_0$, then, since $\mathscr{L} = (a)$, b is in $\mathrm{ad}\, \mathscr{L}$; but being a polynomial without constant term in $\mathrm{ad}\, l_0$, we have $b\Gamma \subseteq \Gamma$ and thus $b \in \mathrm{ad}\, \mathscr{L}_0$, proving our statement. From this we conclude, as in 4.3. a), that $\mathscr{L}_0 = (a)$.

b) We write $G_0 = \exp \mathscr{L}_0$ and show, that G_0 contains the stabilizer S of f. To this end, let us denote by $\{c', g'\}$ a base, dual to $\{c, g\}$, in J'; we observe, that the projection of f onto J' is a nonzero multiple of c'. We write $\tau(a) = \rho(a) \mid J'$ ($a \in G$); obviously S is contained in the stabilizer S_1 of c' with respect to τ, and thus it suffices to show, that $S_1 = G_0$. We have for the matrix expression of $d\tau(l)$ with respect to $\{c', g'\}$

$$d\tau(l) = \begin{pmatrix} 0 & 0 \\ -\gamma(l) & -\delta(l) \end{pmatrix} \qquad (l \in \mathscr{L})$$

from where we conclude, that the orbits of τ on J' are simply connected, and hence S_1 is connected and simply connected. But since $d\tau(l)\,c' = -\gamma(l)\,g'$, we have $S_1 = \exp \mathscr{L}_0 = G_0$.

Let S^0 be the stabilizer of $\pi(f)$; we are going to show, that $S^0 = SS_c^0$. We denote by γ the element in $\mathscr{L}'$ determined by $(l, \gamma) = \gamma(l)$ $(l \in \mathscr{L})$. We have then $\rho(a) = \Delta(a)\,\gamma$ for all a in G_0, where Δ is a homomorphism of G_0 into the multiplicative group of positive numbers, and $d\Delta(l_0) = \delta(l_0)$ $(l_0 \in \mathscr{L}_0)$. We have also, for any s in S^0, $\rho(s)f = f + a(s)\,\gamma$, and $a(s) = 0$ if and only if s belongs to S. Let us write W for the subspace, spanned by f and γ, of $\mathscr{L}'$; since γ is orthogonal to J, the dimension of W equals two. W is obviously invariant with respect to the restriction of ρ to S^0. Writing for the part, in W, of the latter ω, we have for the matrix expression of $\omega(s)$ with respect to $\{f, \gamma\}$

$$\omega(s) = \begin{pmatrix} 1 & 0 \\ a(s) & \Delta(s) \end{pmatrix}, \qquad (s \in S^0).$$

We write, as before, $S_c^0 = \exp R^0$ and observe, that there exist an element $\bar r$, satisfying $-(\operatorname{ad} \bar r)'f = \gamma$ and $-(\operatorname{ad} \bar r)'\gamma = 0$, in R^0. In fact, one sees at once, that $g = \bar r$ has all the required properties. But then we have also $\omega(\exp(t\bar r)) \equiv t$ $(t \in R)$, and thus for any s in S^0: $a(\exp(-a(s)\bar r)s) = 0$, implying, that $\exp(-a(s)\bar r)s$ lies in S_1, and therefore $S^0 = SS_c^0$.

c) We assume, that $\not{f} = (A, \mathscr{L}_0)$ with respect to $\pi(f)$ and show, that this implies $\not{f} = (A, \mathscr{L})$ with respect to f. In fact, one sees at once, using b), that conditions I, II, III α) and III β) in 3.1 are fulfilled. In order to prove, that III γ), too, is satisfied, we denote by $\mathfrak{n}_0$ the nilradical of $\mathscr{L}_0$ and show, that $\mathfrak{n}_0 = \mathscr{L}_0 \cap \mathfrak{n}$. It is evident, that the right hand side is contained in $\mathfrak{n}_0$. Therefore, since $\mathscr{L}$ is solvable, it suffices to prove, that if b is any element in $\mathfrak{n}_0$, then we have $[b, \mathscr{L}] \subset \mathscr{L}_0$. But for this it is enough to remark, that $\mathfrak{n}_0$ lies in the centralizer of J, which is an ideal, contained in $\mathscr{L}_0$, of $\mathscr{L}$. We have $\not{f} \cap (\mathfrak{n}_0)_C = \not{f} \cap (\mathscr{L}_0) \cap \mathfrak{n}_C = \not{f} \cap \mathfrak{n}_C$, and therefore to show, that $\not{f} \cap \mathfrak{n}_C$ is maximal self orthogonal with respect to the bilinear form $([x, y], f)$ $(x, y \in \mathfrak{n}_C)$, it suffices to show, that if n in $\mathfrak{n}_C$ satisfies $0 = ([n, h], f)$ for all h in $\not{f} \cap \mathfrak{n}_C$, then n belongs to $(\mathscr{L}_0)_C$. If $\delta \equiv 0$, then $\mathscr{L}_0$ is an ideal in $\mathscr{L}$, and thus $\mathfrak{n} \subset \mathscr{L}_0$; therefore we can assume $\delta \not\equiv 0$. In this case we have $J \subset \mathfrak{n}$. In fact, if l in $\mathscr{L}$ is such, that $\delta(l) = 1$, then $[l, g] = g + \gamma(l)\,c$, and our statement follows from $[\mathscr{L}, \mathscr{L}] \subset \mathfrak{n}$. On the other hand, one sees easily, that $J \subset \not{f}$; consequently $J \subset \not{f} \cap \mathfrak{n}_C$. Therefore, in particular, $0 = ([n, g], f) = \gamma(n)(c, f)$, proving, that $n \in (\mathscr{L}_0)_C$.

d) We have, by virtue of the second half of b) $A^0 = S^0D = SS_c^0D = SD = A$. Therefore, if $T = \mathrm{ind}(\not{f}, f, \chi)$, we can form the irreducible representation $U = \mathrm{ind}(\not{f}, f_0, \chi)$ $(f_0 = \pi(f))$ of G_0; it belongs to the orbit $O_0 \subset \mathscr{L}_0'$. Reasoning, as in 4.3. e) we prove, that $T = \mathrm{ind}_{G_0 \uparrow G} U$.

e) Let us show, that $\pi^{-1}(O_0) = O_1$ is the G_0 orbit, containing f, in $\mathscr{L}'$. Using the notations of b), for any l in O_1 there exist an a in G_0, such that $l = \rho(a)f + K\gamma = \rho(a)(f + K\Delta(a^{-1})\gamma) = \rho(as)f$, where $\exp(t\bar{r}) = s$ is in S^0, and $t = K\Delta(a^{-1})$, which proves our statement. If O is closed, we show, using the Remark in 4.3. f) and a) above, that O is closed, which implies, that O_0, too, is closed.

4.7. We assume now, that there exist no abelian ideals, of a dimension not exceeding two and different from the center, but that there is an abelian ideal J, of dimension 3, and having a base $\{j_1, j_2, c\}$, where c is in the center, such that $\mathrm{ad}\, lj_1 = \alpha(l)\, j_2 + \lambda_1(l)c$, and $\mathrm{ad} lj_2 = -\alpha(l)\, j_1 + \lambda_2(l)\, c$ for all l in $\mathscr{L}$. Our assumptions imply $\alpha \not\equiv 0$, and $(c, f) \neq 0$ (we assume $(c, f) = 1$ in the sequel), moreover replacing, if necessary, j_k by $j_k + a_k c$, where the a_k's $(k = 1, 2)$ are suitable chosen constants, we can achieve, that $(j_k, f) = 0$ $(k = 1, 2)$. Let us observe, that the subspace $\{\alpha, \lambda_1, \lambda_2\}$ of $\mathscr{L}'$ is of dimension 3. In fact, if $\dim\{\alpha, \lambda_1, \lambda_2\} = 1$, there exist constants a_k such that $\lambda_1 = a_1\alpha$ and $\lambda_2 = -a_2\alpha$. But then $j_1 + a_2 c$ and $j_2 + a_1 c$ span an abelian ideal of dimension 2. Next we show, that $\dim\{\alpha, \lambda_1, \lambda_2\} = 2$ implies $\alpha \equiv 0$, and thus gives a contradiction. In fact, let us denote by $\mathscr{B}$ the collection of all 3×3 matrices, with real coefficients, satisfying the following conditions: $a_{21} = -a_{12}$, a_{21}, a_{31} and a_{32} are arbitrary and the remaining coefficients are zero. $\mathscr{B}$ is a Lie algebra of dimension 3, having only one subalgebra of dimension 2, which is determined by $0 = a_{21} = a_{12}$. Using $\{j_1, j_2, c\}$ as a base, we have for any l in $\mathscr{L}$

$$\mathrm{ad}\, l \mid J = \begin{pmatrix} 0 & -\alpha(l) & 0 \\ \alpha(l) & 0 & 0 \\ \lambda_1(l) & \lambda_2(l) & 0 \end{pmatrix}.$$

Consequently, if $\dim\{\alpha, \lambda_1, \lambda_2\} = 2$, $\mathrm{ad}\, \mathscr{L} \mid J$ is identifiable to a 2-dimensional subalgebra of $\mathscr{B}$, and therefore $\alpha \equiv 0$, proving our statement.

a) We write $\mathscr{L}_0 = \ker \lambda_1 \cap \ker \lambda_2$; $\mathscr{L}_0$ is a subalgebra of $\mathscr{L}$, and by virtue of what we have just seen, $\dim \mathscr{L}_0 + 2 = \dim \mathscr{L}$. Let us denote by Γ the two-dimensional subalgebra spanned by j_1 and j_2. We observe, that $\mathrm{ad}\, \mathscr{L}_0 = \{a;\ a \in \mathrm{ad}\, \mathscr{L},\ a\Gamma \subseteq \Gamma\}$, from which, as in

4.6. a) we conclude, that ad $\mathscr{L}_0$ is an algebraic subalgebra of $L(\mathscr{L})$, and that $\mathscr{L}_0 = (a)$.

b) Let us write $G_0 = \exp \mathscr{L}_0$, and show, that G_0 contains the stabilizer S of f. With notations, analogous to those of 4.6. b) one has, by virtue of $\dim\{\alpha, \lambda_1, \lambda_2\} = 3$

$$\tau(G_0) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ x_1 & \cos\varphi & -\sin\varphi \\ x_2 & \sin\varphi & \cos\varphi \end{pmatrix} ; \varphi, x_1, x_2 \in R \right\} .$$

Hence, in particular, $\tau(G_0)\, c' = (c' + x_1 j_1' + x_2 j_2';\ x_1, x_2 \in R\}$, and thus S_1 is connected and simply connected. But we have also $d\tau(l)\, c' = -(\lambda_1(l)\, j_1' + \lambda_2(l)\, j_2')\ (l \in \mathscr{L})$, whence we conclude, that $S_1 = \exp \mathscr{L}_0 = G_0$, and thus $S \subset G_0$.

Let us prove now, that $S^0 = SS_c^{\,0}$. Let λ_k be the element of $\mathscr{L}'$ determined by $(l, \lambda_k) = \lambda_k(l)\ (l \in \mathscr{L},\ k = 1, 2)$; we have for any l_0 in $\mathscr{L}_0 : -(\operatorname{ad} l_0)'\lambda_1 = \alpha(l_0)\, \lambda_2,\ -(\operatorname{ad} l_0)'\lambda_2 = -\alpha(l_0)\, \lambda_1$. We denote by W the subspace, spanned by f, λ_1 and λ_2, of $\mathscr{L}'$. Observe, that $\dim W = 3$, since otherwise f would be a linear combination of λ_1 and λ_2, and thus orthogonal to J. We have for any s in $S^0 : \rho(s)f = f + a_1(s)\, \lambda_1 + a_2(s)\, \lambda_2$, and s belongs to S if and only if $a_k(s) = 0$ $(k = 1, 2)$. In this fashion, W is invariant under the restriction of ρ to S^0; we write $\omega(s)$ for the part of the latter in W. Let A be the homomorphism, satisfying $dA = \alpha$, of G into the additive group of the reals. Then the matrix expression, with respect to the base $\{f, \lambda_1, \lambda_2\}$, of $\omega(s)$ is given by

$$\omega(s) = \begin{pmatrix} 1 & 0 & 0 \\ a_1(s) & \cos A(s) & -\sin A(s) \\ a_2(s) & \sin A(s) & \cos A(s) \end{pmatrix}, \qquad (s \in S^0).$$

Let us observe, that putting $\bar r_k = j_k$ we have $-(\operatorname{ad} \bar r_k)'f = \lambda_k$ and $(\operatorname{ad} \bar r_k)'\lambda_i = 0\ (i, k = 1, 2)$. In this fashion we obtain

$$\omega(\exp(t\bar r) \exp(t\bar r)) = \begin{pmatrix} 1 & 0 & 0 \\ t_1 & 1 & 0 \\ t_2 & 0 & 0 \end{pmatrix}, \qquad (t_1, t_2 \in R).$$

Therefore, for any s in S^0, the element $\exp(-a(s)\bar r_1) \exp(-a_2(s)\bar r_2)\, s$ lies in S_1, proving, that $S^0 = S \,.\, S_c^{\,0}$.

c) If $\not f = (A, \mathscr{L}_0)$ with respect to $\pi(f)$, then we have also $\not f = (A, \mathscr{L})$ with respect to f. In fact, by virtue of b) above, it is clear, that conditions I, II, III α) and III β) in 3.1 are fulfilled. In order to verify, that condition III γ), too, is satisfied, let us observe, that J is contained in the nilradical $\mathfrak{n}$ of $\mathscr{L}$. In fact, this follows at once from $\alpha \not\equiv 0$ and $[\mathscr{L}, \mathscr{L}] \subset \mathfrak{n}$. From here we obtain the desired conclusion through an easy modification of the argument of 4.6. c).

d) By virtue of the second half of b) we have again $A^0 = S^0 D = S \,.\, S_c{}^0 \,.\, D = SD = A$. Therefore, we can form U as in 4.6. d) and we have again $T = \operatorname{ind}_{G_0 \uparrow G} U$.

e) We prove next, that $\pi^{-1}(O_0) = O_1$ is the G_0 orbit, containing f, in $\mathscr{L}'$. With the notations of b), we have for any l in O_1: $l = \rho(a)f + c_1\lambda_1 + c_2\lambda_2$ with a in G_0, and thus $l = \rho(a)(f + c_1'\lambda_1 + c_2'\lambda_2) = \rho(as)f$, where $s = \exp(c_1'\bar{r}_1)\exp(c_2'\bar{r}_2)$ and $c_1' = \cos A(a)\, c_1 + \sin A(a)\, c_2$, $c_2' = -\sin A(a)\, c_1 + \cos A(a)\, c_2$, proving our statement. If O is closed then, by virtue of a), O_0, too, is closed (cf. 4.3. f)).

4.8. Let us assume now, that if J is an abelian ideal, different from the center and satisfying $\dim J \leqslant 3$, then we have $\dim J = 3$, and J has a base $\{j_1, j_2, c\}$, where c is in the center, such that $\operatorname{ad} l j_1 = \alpha_1(l)\, j_1 + \alpha_2(l)\, j_2 + \lambda_1(l)\, c$, $\operatorname{ad} l j = -\alpha_2(l)\, j_1 + \alpha_1(l)\, j_2 + \lambda_2(l) c$, and $\alpha_1 \not\equiv 0$, $\alpha_2 \not\equiv 0$. Our assumptions imply, that $(c, f) \neq 0$, and as in 4.7, we can assume, that $(j_k, f) = 0$ $(k = 1, 2)$ and $(c, f) = 1$. Let us observe, that α_1 and α_2 are linearly independent. In fact, since $\operatorname{ad} \mathscr{L}$ is an algebraic subalgebra of $L(\mathscr{L})$, $\operatorname{ad} \mathscr{L} \mid J$ is algebraic in $L(J)$, and therefore the direct sum of the underlying spaces of the ideal of nilpotent elements and of an abelian algebraic subalgebra $\mathfrak{a}$ of semi-simple endomorphisms (cf. 2.1. a)). The roots of the component in $\mathfrak{a}$, of $\operatorname{ad} l$ $(l \in \mathscr{L})$ are $\alpha_1(l) \pm i\alpha_2(l)$, and thus repeating a reasoning of 4.5. a) we can prove, that α_1 and α_2 cannot be proportional. We are going to show now, that even $\dim\{\alpha_1, \alpha_2, \lambda_1, \lambda_2\} = 4$. To this end, let us denote by $\mathscr{B}$ the Lie algebra of all 3×3 matrices with real coefficient subject to the following conditions: $a_{11} = a_{22}$, $a_{21} = -a_{12}$, $a_{31} = a_{32} = 0$. We write e_1, e_2, e_3 and e_3 for the elements of $\mathscr{B}$, the nonvanishing coefficients of which are given by $a_{11} = 1$, $a_{21} = 1$, $a_{13} = 1$ and $a_{23} = 1$ respectively. The system $\{e_k;\ k = 1, 2, 3, 4\}$ forms a base in $\mathscr{B}$, and the nonvanishing brackets are $[e_1, e_3] = e_3$, $[e_1, e_4] = e_4$, $[e_2, e_3] = e_4$ and $[e_2, e_4] = -e_3$. Let us set $\psi(l) = -(\operatorname{ad} l)' \mid J'$ $(l \in \mathscr{L})$. The operator $\psi(l)$ on $L(J')$, when expressed as a matrix with respect to a base, dual to $\{j_1, j_2, c\}$, in J', can be viewed as the element $-\alpha_1(l)e_1 + \alpha_2(l)e_2 - \lambda_1(l)e_3 - \lambda_2(l)e_4$ of $\mathscr{B}$, and the map $l \to \psi(l)$ is a homomorphism of $\mathscr{L}$ into $\mathscr{B}$. Let us show now, that $\dim\{\alpha_1, \alpha_2, \lambda_1, \lambda_2\} = 2$ is impossible. One verifies easily, that if the elements $f_1 = e_1 + a_1e_3 + a_2e_4$ and $f_2 = e_2 + b_1e_3 + b_2e_4$ form a subalgebra of $\mathscr{B}$, then we have $a_1 = b_2 = a$ and $a_2 =$

$-b_1 = b$. Therefore, if our statement is false, that is, if $\dim \psi(\mathscr{L}) = 2$, by virtue of $\dim\{\alpha_1, \alpha_2\} = 2$ we have relations of the form $\lambda_1 = a\alpha_1 + b\alpha_2$ and $\lambda_2 = b\alpha_1 - a\alpha_2$. But this implies, that the elements $j_1 + ac$ and $j_2 + bc$ span an abelian ideal, of dimension two, in $\mathscr{L}$, contrary to our starting assumption. But $\dim\{\alpha_1, \alpha_2, \lambda_1, \lambda_2\} = 3$, too, is impossible. In order to see this, it is enough to remark, that the elements $(e_1 + ae_4, e_2 + be_4, e_3 + ce_4\}$ or $\{e_1 + ae_3, e_2 + be_3, e_4 + ce_3\}$ cannot span subalgebras of $\mathscr{B}$.

a) Let us write $\mathscr{L}_0 = \ker \lambda_1 \cap \ker \lambda_2$; $\mathscr{L}_0$ is a subalgebra, of codimension two, of $\mathscr{L}$. Proceeding as in 4.7. a), one can prove, that ad $\mathscr{L}_0$ is algebraic in $L(\mathscr{L})$ and consequently $\mathscr{L}_0 = (a)$.

b) Let us set $G_0 = \exp \mathscr{L}_0$. One shows, reasoning as in 4.7. b), That G_0 contains S. In fact, we have here, since $\dim\{\alpha_1, \alpha_2, \lambda_1, \lambda_2\} = 4$,

$$\tau(G_0) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ x_1 & e^t \cos\varphi & -e^t \sin\varphi \\ x_2 & e^t \sin\varphi & e^t \cos\varphi \end{pmatrix}; \qquad t, \varphi, x_1, x_2 \in R \right\}.$$

We have also $S^0 = S \,.\, S_c^0$. This can be proved by adopting the argument of 4.7. b), and in what follows we confine ourselves to indicate the necessary modifications. We have now for any l_0 in $\mathscr{L}_0$: $-(\text{ad } l_0)'\lambda_1 = \alpha_1(l_0)\lambda_1 + \alpha_2(l_0)\lambda_2$ and $-(\text{ad } l_0)'\lambda_2 = -\alpha_2(l_0)\lambda_1 + \alpha_1(l_0)\lambda_2$. Putting $\bar{r}_k = j_k$ we have also $-(\text{ad } \bar{r}_k)'f = \lambda_k$, $(\text{ad } \bar{r}_i)'\lambda_k = 0$ $(i, k = 1, 2)$, and therefore

$$\omega(\exp(t_1\bar{r}_1)\exp(t_2\bar{r}_2)) = \begin{pmatrix} 1 & 0 & 0 \\ t_1 & 1 & 0 \\ t_2 & 0 & 1 \end{pmatrix},$$

and from here we obtain the desired conclusion as in 4.7. b).

c) Since $\alpha_1 \not\equiv 0$, $\alpha_2 \not\equiv 0$, we conclude, that J is contained in the nilradical $\mathfrak{n}$ of $\mathscr{L}$. Taking into account this, we prove, that $f = (A, \mathscr{L}_0)$ implies $f = (A, \mathscr{L})$ as in 4.6. c).

d) Proceeding as in 4.7. d) and e), we construct a unitary representation U, belonging to O_0, of $G_0 = \exp \mathscr{L}_0$, such that $T = \text{ind}_{G_0\uparrow G}\, U$, and show, using a) above, that O_0 is closed, if so is O_0.

4.9. We sum up the discussion of 4.1–4.8 as follows. If $\mathscr{L}$ contains a nonzero ideal J, orthogonal to f, then it has also an ideal of this kind contained in the nilradical $\mathfrak{n}$. In fact, to see this, it suffices to take a nontrivial minimal ideal in J and observe, that any such ideal belongs to $\mathfrak{n}$. Thus if there exist at all nontrivial ideals, orthogonal to f, then $\mathscr{L}$ contains a nonzero ideal, satisfying the condition of 4.1.

If no such ideals exist, then the center C of $\mathscr{L}$ is either trivial, or of dimension one, and in the latter case it is not orthogonal to f. We have the following two possibilities: There are minimal nonzero ideals, different from C, or not. If yes, then we have one of the cases 4.3–4.5; to satisfy the conditions of 4.2, in the case of 4.3 we can take $G = G_0$ and $U = V$, in the case of 4.4 or 4.5 G_1/G_0, if not trivial, is infinite cyclic. If there exist no minimal ideals different from C, but there are abelian ideals properly containing C, then we have also an ideal J of this sort such that dim $J/C = 1$ or 2, and the action of the adjoint representation of on J/C is irreducible. We have then one of the cases 4.6–4.8, and we can always take in 4.2 $G = G_0$ and $U = V$. Summing up, if either $\mathscr{L}$ possesses a nonzero ideal orthogonal to f, or there is an abelian ideal, different from the center (or both), then we can find an ideal satisfying the conditions of 4.1 or 4.3–4.8.

For later use (cf. 5.6) we observe, that in cases 4.3–4.8 (cf. b) loc. cit. each time), we can find an element $\bar{r}$ (or elements $\bar{r}_1$, $\bar{r}_2$ respectively with $[\bar{r}_1, \bar{r}_2] = 0$ if codim $\mathscr{L}_0 = 2$) in $\mathscr{L}_0$, such that $\rho(\exp(t\bar{r}))f = f + l't$ $(t \in R)$ (or $\rho(\exp(t_1\bar{r}_1)\exp(t_2\bar{r}_2))f = f + l_1't_1 + l_2't_2$; t_1, t_2R), where $l'(l_1', l_2'$ respectively) spans $\mathscr{L}_0^{\perp}$.

4.10. Let us assume, that the previous conditions cannot be fulfilled. Then dim $C = 1$, and if c generates C, we have $(c, f) \neq 0$; moreover any abelian ideal of $\mathscr{L}$ coincides with C. Writing $\mathscr{L} = \mathfrak{n} + \mathfrak{a}$ (cf. 2.1. b)), we are going to show, that for any a in $\mathfrak{a}$, the eigenvalues of ad(a) are purely imaginary (for this cf. [*10*], 5.12). To this end, let us set $\mathfrak{n}' = \mathfrak{n}/C$, and denote by φ the canonical homomorphism from $\mathfrak{n}$ onto $\mathfrak{n}'$. We write $\mathfrak{n}' = \sum_{j=1}^{m} \mathfrak{n}_j'$, where $\mathfrak{n}_j'$ is minimal invariant under ad $\mathfrak{a}$, and consequently dim $\mathfrak{n}_j' = 1$ or 2 $(j = 1, 2, \ldots, m)$. Let us show, that dim $\mathfrak{n}_j' = 1$ is impossible. In fact, in this case, if n is an element in $\mathfrak{n} - C$, such that $\varphi(n)$ lies $\mathfrak{n}_j'$, c and n span a two-dimensional abelian ideal, contrary to our assumptions. Since ad $\mathfrak{a}$ is algebraic in $L(\mathscr{L})$, the restriction of ad $\mathfrak{a}$ to $\mathfrak{n}_j'$ is algebraic in $L(\mathfrak{n}_j')$. Therefore, using an argument employed at the begin of 4.5, we conclude, that if, for a in $\mathfrak{a}$, ad(a) has a root, which is not purely imaginary, then for some $j = 1, 2, \ldots, m$ and b in $\mathfrak{a}$, ad$(b) \mid \mathfrak{n}_j'$ is the identity operator on $\mathfrak{n}_j'$. Let n_1 and n_2 be elements of μ, such that $\varphi(n_1)$ and $\varphi(n_2)$ span $\mathfrak{n}_j'$. We have $[n_1, n_2] \neq 0$ since otherwise n_1, n_2 and c would span a three-dimensional abelian ideal of $\mathscr{L}$. We can obviously assume $[n_1, n_2] = c$. But then

$$0 = (\mathrm{ad}(a))\, c = (\mathrm{ad}(a))[n_1, n_2] = [(\mathrm{ad}(a))\, n_1, n_2] + [n_1, (\mathrm{ad}(a))\, n_2] = 2c,$$

and thus $c = 0$, giving a contradiction.

We know (cf. 2.2. c)), that $\mathfrak{n}$ is spanned by $2n + 1$ elements $\{p_k, q_j;\ k, j = 1, 2, \ldots, n\}$ and c, satisfying the relations $[p_k, q_j] = \delta_{kj}c$, all other brackets being zero. In this fashion we can conclude, that if $\mathscr{L}$ satisfies our starting assumption, then it is a special algebra (cf. 1), that is $\mathscr{L} = \mathfrak{n} + \mathfrak{a}$, where $\mathfrak{n}$ is as just described, and $\mathrm{ad}(a)$ $(a \in \mathfrak{a})$ has only purely imaginary roots.

5.1. Before proceeding, we state again the main theorem, but in a form, which is slightly more general, than that given in 1. Let G be a Lie group such that $G = (a)$ (cf. 2.1. c)) and $\mathscr{L}$ its Lie algebra. We denote by $\mathscr{F}$ the collection of all roots (cf. 1) of $\mathscr{L}$, and consider the open subdomain $\tilde{B}$, defined as $\{l;\ \mathrm{Im}\,\alpha(l) \neq 2\pi n,$ $n = \text{integer} \neq 0,\ \alpha \in \mathscr{F}\}$, of the underlying space of $\mathscr{L}$. For α in $\mathscr{F}$ let A be the homomorphism, determined by $dA = \mathrm{Im}\,\alpha$, of G into the additive group of the reals, and let us write $\tilde{D} = \{g;\ A(g) \neq 2\pi m,$ $m = \text{integer} \neq 0,\ \alpha \in \mathscr{F}\}$. We recall (cf. [9]. Theorem 2, p. 119), that the restriction of the exponential map to $\tilde{B}$ establishes an analytic isomorphism with $\tilde{D}$. We set $B = \{l;\ |\mathrm{Im}\,\alpha(l)| < 2\pi,\ \alpha \in \mathscr{F}\}$; B is the connected component, containing the neutral element of $\mathscr{L}$, of $\tilde{B}$; we write $D = \exp B$. Let T be a unitary representation of G. We write $P(T)$ for the collection of all complex valued C_c^∞ functions on G, the support of which is compact and lies in D, and for which the operator $T(\varphi) = \int_G \varphi(a)\, T(a)\, da$, where da is a fixed right invariant measure on G, satisfies $T(\varphi) \geqslant 0$. Given any C_c^∞ function φ, the support of which is contained in D, on G, there exist a unique C_c^∞ function ψ, with support in B, on $\mathscr{L}$ such that $\psi(l) \equiv \varphi(\exp l)$ $(l \in \mathscr{L})$. In the following we shall often write $\varphi(l)$ for $\psi(l)$. With these notations we have the following

THEOREM. *Let T be an irreducible unitary representation, belonging to the closed orbit O in $\mathscr{L}'$, of G, and let φ be an element of $P(T)$. Then the operator $T(\varphi)$ is of trace class, and for its trace we have the following expression*:

$$\mathrm{Tr}(T(\varphi)) = \int_0 \hat{\omega}(l')\, dv,$$

$\hat{\omega}(l')$ is the Fourier transform, over $\mathscr{L}$, of the function $\omega(l) = \mu(l)\varphi(l)$, where $\mu(l)$, which does not depend on φ, is given by

$$\mu(l) = (\Delta(\exp l))^{1/2} \prod_{\alpha \in \mathscr{F}_1} \frac{\exp(\frac{1}{2}\alpha(l)) - \exp(-\frac{1}{2}\alpha(l))}{\alpha(l)} \prod_{\beta \in \mathscr{F}_2} \frac{\exp \beta(l) - 1}{\beta(l)}\,.$$

Here $\mathscr{F}_1$ and $\mathscr{F}_2$ are disjoint subfamilies of $\mathscr{F}$, Δ is given by $d(a_0a) = \Delta(a_0)\, da$ $(a_0 \in G)$, and dv is a positive invariant measure on O.

Remark. Our proof will show, that dv and $\mu(l)$ can be chosen to be the same for any T belonging to O.

We shall discuss an algorithm to compute dv, which we shall call the canonical measure belonging to 0, later (cf. 5.6 and 6.6).

COROLLARY. *Let T be as above, and let φ be a C_c^∞ function on G. Then the operator $T(\varphi)$ is of class Hilbert-Schmidt.*

Proof. Let us observe first, that it is enough to prove our statement under the assumption, that the support of φ is contained in a fixed neighborhood of the identity U. Let us choose U such, that $U = U^{-1}$, and the closure of U_2 is compact and lies in D. Then if the support of the C_c^∞ function φ is contained in U, the convolution ψ of $\varphi(a)$ and of $\varphi(a^{-1})/\Delta(a)$ belongs to $P(T)$. Since by virtue of the theorem $T(\psi) = T(\varphi)\, T^*(\varphi)$ is of trace class, we have the desired conclusion.

We shall prove the theorem by induction, proceeding according to the dimension of G. Since, if $\dim G = 1$, the statement is clear, we assume in the following, that $\dim G > 1$.

In the remainder of this section, we shall set up relations between the traces corresponding to representations T and U, connected with each other as in 4.1 and 4.2. This will finish the proof up to the computation of the trace for certain irreducible representations of special groups; this will be done in the next section.

5.2. Let us assume first, that there exist a nonzero ideal J, orthogonal to an element f of O, of $\mathscr{L}$; in what follows we shall assume that J is a minimal ideal. Let us write $\tilde{G} = G/\exp J$ and $\tilde{\mathscr{L}} = \mathscr{L}/J$. We denote the canonical homomorphism from G onto $\tilde{G}$ by Ψ, and recall (cf. 4.1), that there exist an irreducible unitary representation U, belonging to the orbit $O \subset \tilde{\mathscr{L}}' = J^\perp$, of G which satisfies $T = U \circ \Psi$. In the following we shall distinguish notions, relative to $\tilde{G}$, in most cases by $\tilde{}$.

a) Let φ be an element of $P(T)$ (cf. 5.1) and let us show, that there exist a φ_1 in $P(U)$, such that $U(\varphi_1) = T(\varphi)$. To this end, let us choose an invariant measure dj on J, and a right invariant measure $d\tilde{g}$ on $\tilde{G}$, such that we have

$$\int_G f(a)\, da = \int_{\tilde{G}} \left(\int_J f(\exp j \cdot g)\, dj\right) d\tilde{g}$$

for all f which is continuous and of a compact support on G. Then we have

$$T(\varphi) = \int_G \varphi(a)\, T(a)\, da = \int_{\tilde{G}} U(\tilde{g})\Big(\int_J \varphi(\exp j \cdot g)\, dj\Big)\, d\tilde{g} = U(\varphi_1)$$

where $\varphi_1(\tilde{g}) = \int_J \varphi(\exp j \,.\, g)\, dj$ ($\tilde{g} = \Psi(g)$). Therefore, to obtain the desired conclusion it is enough to prove, that the support of φ_1 is contained in $\tilde{D}$. Suppose, that $\varphi_1(g) \neq 0$ for $\tilde{g} = \exp \tilde{l} = \Psi(g)$, where $g = \exp l$. Then for some j in J $\exp j \,.\, g = \exp(l + j_1)$ ($j_1 \in J$) belongs to the support of φ, and thus, writing α_1 for the imaginary part of α : $|\alpha_1(l)| = |\alpha_1(l + j_1)| < 2\pi - \epsilon$ ($\epsilon > 0$) for any α in $\mathscr{F}$, where ϵ depends on φ only. But if $\tilde{\alpha}$ is a root of $\tilde{\mathscr{L}}$, $\tilde{\alpha} \circ \psi = \alpha$ ($\psi = d\Psi$) is a root of $\mathscr{L}$, and in this fashion $|\tilde{\alpha}_1(\tilde{l})| = |\alpha_1(\psi(l))| = |\alpha_1(l)| < 2\pi - \epsilon$, proving our statement.

b) Since J is a minimal ideal, we have the following two possibilities. I. $\dim J = 1$, $J = \{j\}$ and $\operatorname{ad} lj = \lambda(l)\, j$, II. $\dim J = 2$, $J = \{j_1, j_2\}$ and $\operatorname{ad} lj_1 = \alpha_1(l)\, j_1 + \alpha_2(l)\, j_2$, $\operatorname{ad} lj_2 = -\alpha_2(l)\, j_1 + \alpha_1(l)\, j_2$ ($l \in \mathscr{L}$) where $\alpha_2 \not\equiv 0$; λ and $\alpha = \alpha_1 \pm i\alpha_2$ are roots of $\mathscr{L}$. Let l be a fixed element in $\mathscr{L}$; an elementary computation, the details of which we omit, yields the following relations I. $\exp j \cdot \exp l = \exp(l + [\lambda(l)/(\exp \lambda(l) - 1)]j)$, II. $\exp(x_1 j_1 + x_2 j_2) \exp l = \exp(l + y_1 j_1 + y_2 j_2)$, where $y_1 + iy_2 = (x_1 + ix_2)/[(\exp(\alpha(l)) - 1)/\alpha(l)]$. (Note, that the value of $[\exp(t) - 1]/t$ at $t = 0$ is 1.) By virtue of these identities, making use of a notational convention introduced at the beginning of 5.1, we obtain

$$\varphi_1(\tilde{l}) = [(\exp(\lambda(l)) - 1)/\lambda(l)] \int_J \varphi(l + j)\, dj, \qquad (\tilde{l} = \psi(l))$$

or

$$\varphi_1(\tilde{l}) = |(\exp(\alpha(l)) - 1)/\alpha(l)|^2 \int_J \varphi(l + j)\, dj,$$

according to whether $\dim J = 1$ or 2 respectively.

c) Since $\dim \tilde{G} < \dim G$ and $\tilde{G} = (a)$ (cf. 4.1), by virtue of the assumption of our inductive procedure, we have

$$\operatorname{Tr}(U(\varphi_1)) = \int_O \hat{\omega}_1(\tilde{l}')\, dv.$$

Here $\hat{\omega}_1$ is the Fourier transform, over $\tilde{\mathscr{L}}$, of $\omega_1(\tilde{l}) = \tilde{\mu}(\tilde{l})\, \varphi_1(\tilde{l})$, where $\tilde{\mu}$ is of the form, with respect to $\tilde{\mathscr{L}}$, as μ is in 5.1.

In what follows, we assume, that $\dim J = 2$, and leave the modifications, necessary to settle the case $\dim J = 1$, to the reader.

Let us set $\mu(l) = \tilde{\mu}(\psi(l)) \mid (\exp \alpha(l) - 1)/\alpha(l)|^2$, and observe, that $\mu(l)$ is as in 5.1. In fact, to this end it suffices to note, that $\Delta(\exp l) = \exp(\mathrm{Tr}(\mathrm{ad}\, l))$, and $\mathrm{Tr}(\mathrm{ad}\, l) = 2 \operatorname{Re} \alpha(l) + \mathrm{Tr}(\mathrm{ad}\, \psi(l))$ $(l \in \mathscr{L})$, and therefore

$$(\tilde{\Delta}(\exp \psi(l)))^{1/2} \,|(\exp \alpha(l) - 1)/\alpha(l)|^2 = (\Delta(\exp l)) \frac{e^{\alpha/2} - e^{-\alpha/2}}{\alpha} \frac{e^{\bar{\alpha}/2} - e^{-\bar{\alpha}/2}}{\bar{\alpha}}$$

where α and $\bar{\alpha}$ are elements in $\mathscr{F}$. Since obviously $\mu(l + j) = \mu(l)$ $(j \in J)$, writing $\omega(l) = \mu(l)\, \varphi(l)$, we obtain $(\tilde{l} = \psi(l),\ l \in \mathscr{L})$

$$\omega_1(\tilde{l}) = \tilde{\mu}(\tilde{l})\, \varphi_1(\tilde{l}) = \mu(l) \int_J \varphi(j + l)\, dj = \int_J \omega(j + l)\, dj.$$

Let us observe now, that if V is a finite dimensional real vector space, W a subspace, $\tilde{V} = V/W$, f a C_c^∞ function on V and

$$f_1(v) = \int_W f(v + w)\, dw$$

where dw is a positive translation invariant measure on W, and if we denote by $\hat{f}$ and $\hat{f}_1$ the Fourier transform of f and f_1 over V and $\tilde{V}$ respectively, then with an appropriately chosen dw we have $\hat{f}_1(\tilde{v}') = \hat{f}(\tilde{v}')$ $(\tilde{v}' \in W^\perp = \tilde{V}')$. Applying this to the case, when $V = \mathscr{L}$, $W = J$, $f(l) \equiv \omega(l)$ $(l \in \mathscr{L})$, we get $\hat{\omega}_1(\tilde{l}') \equiv \hat{\omega}(\tilde{l}')$ $(\tilde{l}' \in J^\perp = \tilde{\mathscr{L}}')$. Therefore, finally

$$\mathrm{Tr}(T(\varphi)) = \mathrm{Tr}(U(\varphi_1)) = \int_O \hat{\omega}(l')\, dv$$

as in the Theorem. Since, for any a in G, $\tilde{\rho}(\Psi(a)) = \rho(a) \mid J^\perp$ (cf. 4.1), dv is invariant with respect to ρ.

5.3. We assume next, that there exist no nonzero ideal, orthogonal to f in O, of $\mathscr{L}$, but that there is a nonzero abelian ideal, different from the center. Then (cf. 4.9) we have an ideal J, satisfying the conditions of one of 4.3–4.8. These can be divided into two groups, according to whether, with the notations of 4.2, $G_1 = G_0$ always (cf. 4.3, 4.6–4.8), or possibly $G \neq G_0$ (cf. 4.4–4.5). In what follows we give a full discussion only of 4.8 and of 4.5 (for the latter cf. 5.4 below), and leave the easy modifications necessary to settle the remaining cases to the reader.

The subsequent analysis follows the line of reasoning in Part 3 of [*17*].

a) Let J be an ideal as in 4.8. Then we have a base $\{j_1, j_2, c\}$ in J, such that

$$\mathrm{ad}l j_1 = \alpha_1(l) j_1 + \alpha_2(l) j_2 + \lambda_1(l) c,$$

$$\mathrm{ad}l j_2 = -\alpha_2(l) j_1 + \alpha_1(l) j_2 + \lambda_2(l) c \ (l \in \mathscr{L}),$$

and $\dim\{\alpha_1, \alpha_2, \lambda_1, \lambda_2\} = 4$. Consequently, there exist elements l_1, l_2 in $\mathscr{L}$ satisfying $\lambda_i(l_k) = \delta_{ik}$, $\alpha_i(l_k) = 0$ $(i, k = 1, 2)$. Writing $g_j(t) = \exp(tl_j)$ $(j = 1, 2)$, we have $\sigma(g_j(t)) j_k = j_k + \delta_{jk} t$. From this we conclude, that $\Gamma = \{g_1(t)_1 g_2(t_2);\ t_1, t_2 \in R\}$ is closed in G, and that the map Ψ from $G_0 \times \Gamma$ ($G_0 = \exp \mathscr{L}_0$, $\mathscr{L}_0 = \ker \lambda_1 \cap \ker \lambda_2$, cf. 4.8) onto G defined by $\Psi(g_0, \gamma) = g_0 \gamma$ is a diffeomorphism. In particular, we can identify the right coset space G/G_0 of G according to G_0 with Γ. Given γ in Γ and a in G we can write $\gamma a = g_0(\gamma a) \,.\, \gamma \bar{a}$ ($g_0(\gamma a) \in G_0$, $\gamma \bar{a} \in \Gamma$), and the factors on the right hand side are uniquely determined. The map $\gamma \to \gamma \bar{a}$ of Γ onto itself coincides, under the above identification, with the right action of a on G/G_0.

Let dg_0 be a right invariant measure on G_0, such that $d(g_0' g_0) = \Delta_0(g_0')\, dg_0$ ($g_0' \in G_0$). We recall (cf. [20], p. 45), that given a homomorphism ψ of G into the multiplicative group of the positive numbers, there exist a relatively invariant measure $d\gamma$, satisfying $d\gamma\bar{a} = \psi(a)\, d\gamma$, on Γ, if and only if we have $\Delta_0(g_0) = \psi(g_0)\, \Delta(g_0)$ for all g_0 in G_0. Under the assumptions as above, such a ψ can readily be found. In fact, if $g_0 = \exp l_0$ ($l_0 \in \mathscr{L}_0$), then

$$\Delta_0(g_0)/\Delta(g_0) = \exp(-\mathrm{Tr}(\mathrm{ad}\, l_0 \mid \mathscr{L}/\mathscr{L}_0));$$

but $\mathrm{Tr}(\mathrm{ad}\, l_0 \mid \mathscr{L}/\mathscr{L}_0) = \mathrm{Tr}((\mathrm{ad}\, l_0)' \mid \mathscr{L}_0^{\perp}) = -2\alpha_1$ (cf. 4.8. b)). Therefore it is enough to take $\psi(a) = \exp(2A(a))$, where A is the homomorphism, determined by $dA = \alpha_1$, of G into the additive group of the reals.

b) We know (cf. 4.8. d)), that there exist an irreducible unitary representation U of G_0, such that $\mathrm{ind}_{G_0 \uparrow G}\, U = T$. Let us denote by $H(T)$ and $H(U)$ the representation space of T and U respectively. By virtue of a) $H(T)$ and T can be described as follows: $H(T)$ is the Hilbert space of equivalence classes of functions, with values in $H(U)$, on Γ, satisfying $\int_\Gamma \|f(\gamma)\|^2\, d\gamma < +\infty$, and $T(a)$ ($a \in G$) is obtained from the map $f(\gamma) \to (\psi(a))^{1/2}\, U(g_0(\gamma a)) f(\gamma\bar{a})$ by taking quotients (we shall write $(T(a) f)(\gamma)$ also for the last function in the sequel).

c) Let φ be a function in $P(T)$ (cf. the begin of 5.1); the operator $T(\varphi)$ is nonnegative. The purpose of the subsequent considerations is to express the finite of infinite trace of the operator $T(\varphi)$ in terms of the traces of integrals, with respect to U, of certain functions in $P(U)$.

Let us assume, that the function f, taking its values in $H(U)$, is strongly continuous and vanishes outside of a compact set of Γ. Then we have

$$(T(\varphi)f)(\gamma) = \int_G \varphi(a)(T(a)f)(\gamma)\,da = \int_G \varphi(a)(\psi(a))^{1/2}\,U(g_0(\gamma a))f(\gamma\bar{a})\,da$$
$$= (1/\Delta(\gamma)(\psi(\gamma))^{1/2}) \int_G (\varphi(\gamma^{-1}a)/(\psi(a))^{1/2})\,U(g_0(a))f(\overline{\gamma\gamma^{-1}a})\,\psi(a)\,da.$$

We recall next (cf. [20], p. 43), that, assuming $d\gamma$ properly normalized, we have

$$\int_G h(g)\,\psi(g)\,dg = \int_\Gamma \left(\int_{G_0} f(g_0\gamma)\,dg_0\right) d\gamma$$

for any complex valued function h, which is continuous and vanishes outside of a compact set of G. Therefore we have

$$(T(\varphi)f)(\gamma) = \int_\Gamma K_\varphi(\gamma, \gamma')f(\gamma')\,d\gamma'$$

where the operator valued kernel $K_\varphi(\gamma, \gamma')$ on $\Gamma \times \Gamma$ is given by

$$K_\varphi(\gamma, \gamma') = (1/[\Delta(\gamma)(\psi(\gamma\gamma'))^{1/2}]) \int_{G_0} (\varphi(\gamma^{-1}g_0\gamma')/(\psi(g_0))^{1/2})\,U(g_0)\,dg_0\,.$$

One shows easily, using $T(\varphi) \geqslant 0$, that for any h in $H(U)$, the function $(K_\varphi(\gamma, \gamma')\,h, h)$ is continuous and positive definite on $\Gamma \times \Gamma$ (cf. p. 116 in [15] for a similar computation). Therefore, in particular, we have $K_\varphi(\gamma, \gamma) \geqslant 0$. On the other hand, for a fix γ,

$$K_\varphi(\gamma, \gamma) = (1/\Delta(\gamma)\,\psi(\gamma)) \int_{G_0} (\varphi(\gamma^{-1}g_0\gamma)(\psi(g_0))^{-1/2})\,U(g_0)\,dg_0$$
$$= (1/\Delta(\gamma)\,\psi(\gamma))\,U(\eta_\gamma)$$

where we put $\eta_\gamma(g_0) = \eta(\gamma^{-1}g_0\gamma)$ $(g_0 \in G_0)$ and $\eta(g) = \varphi(g)(\psi(g))^{-1/2}$ $(g \in G)$. Hence, for any fix γ in Γ, also $U(\eta_\gamma) \geqslant 0$. Let us show, that η_γ belongs to $P(U)$. To this end, by virtue of what we have just seen, it is enough to establish, that the support of η_γ is contained in D_0 (we distinguish notions, relative to G_0, by an index 0). Let us denote by F the support of φ. The support of η_γ being contained in $\gamma F\gamma^{-1} \cap G_0 \subset D \cap G_0$, it suffices to prove the inclusion $D \cap G_0 \subset D_0$.

We write $\hat{\mathscr{F}} = \{A; A : G \to R, dA = \operatorname{Im} \alpha, \alpha \in \mathscr{F}\}$. and observe (cf. 5.1), that $D = \{g; g \in G, \ |A(g)| < 2\pi, \text{ for all } A \in \hat{\mathscr{F}}\}$. We recall (cf. 3.3), that any element of $\mathscr{F}_0$ can be obtained by restricting some element of $\mathscr{F}$ to $\mathscr{L}_0$. This obviously implies an analogous connection between $\hat{\mathscr{F}}_0$ and $\hat{\mathscr{F}}$, of which the relation $D \cap G_0 \subset D_0$ is an evident consequence. Since $\eta_\gamma \in P(U)$, $G_0 = (a)$ and U belongs to the closed orbit $O_0 \subset \mathscr{L}_0'$ (cf. 4.2), by virtue of our inductive assuption $U(\eta_\gamma)$ is of trace class, and

$$0 \leqslant \operatorname{Tr}(U(\eta_\gamma)) = \int_{O_0} \hat{\omega}_\gamma(l_0')\, dv_0 = \Delta(\gamma)\, \psi(\gamma)\, \operatorname{Tr}(K_\varphi(\gamma, \gamma))$$

where $\hat{\omega}_\gamma$ is the Fourier transform, over $\mathscr{L}_0$, of the function $\omega_\gamma(l_0) = \mu_0(l_0)\, \eta_\gamma(l_0)$ $(l_0 \in \mathscr{L}_0)$. On the other hand, one can show (cf. [*15*], p. 120), that

$$\operatorname{Tr}(T(\varphi)) = \int_\Gamma \operatorname{Tr}(K_\varphi(\gamma, \gamma))\, d\gamma$$

in the following sense: if one of the two sides is finite, then so is the other, and the two terms are equal. In this fashion, we have also

$$\operatorname{Tr}(T(\varphi)) = \int_\Gamma (1/\Delta(\gamma)\, \psi(\gamma)) \Big(\int_{O_0} \hat{\omega}_\gamma(l_0')\, dv_0 \Big)\, dy.$$

d) We recall (cf. 5.1), that $\mu_0(l_0)$ $(l_0 \in \mathscr{L}_0)$ is a function of the form

$$\mu_0(l_0) = (\Delta_0(\exp l_0))^{1/2} \prod_{\alpha \in \mathscr{F}_1'} \frac{\exp(\frac{1}{2}\alpha(l_0)) - \exp(-\frac{1}{2}\alpha(l_0))}{\alpha(l_0)} \prod_{\beta \in \mathscr{F}_2'} \frac{\exp \beta(l_0) - 1}{\beta(l_0)}$$

where $\mathscr{F}_j'$ $(j = 1, 2)$ are subfamilies of $\mathscr{F}_0$. Since $\Delta_0(\exp l_0)/\psi(\exp l_0) = \Delta(\exp l_0)$, and since any element of $\mathscr{F}_0$ arises by restricting an element of $\mathscr{F}$ to $\mathscr{L}_0$ (cf. 3.3), $\mu_0(l_0)/(\psi(\exp l_0)^{1/2})$ can be viewed as the restriction to $\mathscr{L}_0$ of a function $\mu(l)$ $(l \in \mathscr{L})$ of the type as in 5.1. We have obviously $\mu(\sigma(a)\, l) = \mu(l)$ for all $a \in G$ and $l \in \mathscr{L}$, and therefore can conclude, that

$$\begin{aligned} \omega_\gamma(l_0) &= \mu_0(l_0)\, \eta_\gamma(l_0) = (\mu_0(l_0)/(\psi(\exp l_0))^{1/2})\, \varphi(\sigma(\gamma^{-1})\, l_0) \\ &= \mu(\sigma(\gamma^{-1})\, l_0)\, \varphi(\sigma(\gamma^{-1})\, l_0) = \omega(\sigma(\gamma^{-1})\, l_0), \end{aligned}$$

where $\omega(l) = \mu(l)\, \varphi(l)$ $(l \in \mathscr{L})$.

e) Our next objective is to express $\hat{\omega}_\gamma(l_0)$ through the Fourier transform, over $\mathscr{L}$, of $\omega(l)$. Given a positive translation invariant measure dl on $\mathscr{L}$, let us determine the measure dl', of the same sort, on $\mathscr{L}'$ in such a fashion, that

$$\hat{\omega}(l') = \int_{\mathscr{L}} \omega(l) \exp[i(l, l')]\, dl$$

imply

$$\omega(l) = \int_{\mathscr{L}'} \hat{\omega}(l') \exp[-i(l, l')]\, dl'.$$

The measure dl' so normalized will be referred to later as the dual of dl. Noting, that $\det \rho(a) = \Delta(a^{-1})$ $(a \in G)$, we have for any l_0 in $\mathscr{L}_0$

$$\begin{aligned} \omega(\sigma(\gamma^{-1})\, l_0) &= \int_{\mathscr{L}'} \hat{\omega}(l') \exp[-i(\sigma(\gamma^{-1})\, l_0\,, l')]\, dl' \\ &= \Delta(\gamma) \int_{\mathscr{L}'} \hat{\omega}(\rho(\gamma^{-1})\, l') \exp[-i(l_0\,, l')]\, dl'. \end{aligned}$$

Let dl_0 be the invariant measure, utilized in forming $\hat{\omega}_\gamma(l_0)$; taking its dual dl_0', we determine the measure $dl_0{}^\perp$ on $\mathscr{L}_0{}^\perp$, such that we have

$$\int_{\mathscr{L}'} h(l')\, dl' = \int_{\mathscr{L}_0'} \left(\int_{\mathscr{L}_0^\perp} h(l' + l_0{}^\perp)\, dl_0{}^\perp \right) dl_0'$$

for any function h, which is continuous and of a compact support on $\mathscr{L}'$. In this fashion we can write

$$\begin{aligned} \omega_\gamma(l_0) = \omega(\sigma(\gamma^{-1})\, l_0) = \Delta(\gamma) \int_{\mathscr{L}_0'} &\left[\int_{\mathscr{L}_0^\perp} \hat{\omega}(\rho(\gamma^{-1})(l' + l_0{}^\perp))\, dl_0{}^\perp \right] \\ &\times \exp[-i(l_0\,, l_0')_0]\, dl_0' \end{aligned}$$

where $(l_0\,, l_0')_0$ stands for the canonical bilinear form on $\mathscr{L}_0 \times \mathscr{L}_0'$, and l_0' is the projection of $l' \in \mathscr{L}'$ on $\mathscr{L}_0'$. From this, using the Fourier inversion formula on $\mathscr{L}_0$, we conclude, that

$$\hat{\omega}_\gamma(l_0') = \Delta(\gamma) \int_{\mathscr{L}_0^\perp} \hat{\omega}(\rho(\gamma^{-1})(l' + l_0{}^\perp))\, dl_0{}^\perp.$$

Therefore, by virtue of the final formula in c)

$$\mathrm{Tr}(T(\varphi)) = \int_\Gamma (1/\psi(\gamma)) \left(\int_{O_0} \left(\int_{\mathscr{L}_0^\perp} \hat{\omega}(\rho(\gamma^{-1})(l' + l_0{}^\perp))\, dl_0{}^\perp \right) dv_0 \right) d\gamma.$$

f) We are going to show, that there is a Borel measure dv, invariant under ρ, on O, such that for any function h, which is continuous and of a compact support on O we have

$$\int_O h(l')\, dv = \int_\Gamma \left(\int_{O_0} \left(\int_{\mathscr{L}_0^\perp} h(\rho(\gamma^{-1})(l' + l_0^\perp))\, dl_0^\perp\right) dv_0\right) d\gamma/\psi(\gamma).$$

To this end, we write again $O_1 = \rho(G_0)f$, and recall (cf. 4.7. e)), that along with O, O_1, too, is closed. Since $\det(\rho(a) \mid \mathscr{L}_0^\perp) = \psi(a)$ for all a in G_0 (cf. a)), there exist a Borel measure dp on O, such that

$$\int_{O_1} h(p)\, dp = \int_{O_0} \left(\int_{\mathscr{L}_0^\perp} h(l' + l_0^\perp)\, dl_0^\perp\right) dv_0$$

and $d(\rho(a)\, p) = \psi(a)\, dp$ $(a \in G_0)$. Let us observe next, that the map ϕ from $\Gamma \times O$ onto O, defined by $\phi(\gamma, p) = \rho(\gamma^{-1})\, p$ $(\gamma \in \Gamma,\ p \in O_1)$ is a homeomorphism. Assuming, that $\gamma = g_1(t_1)\, g_2(t_2)$ (cf. a)), we have $(j_k, \rho(\gamma^{-1})\, p) = (\sigma(\gamma)\, j_k, p) = (j_k, p) + t_k(c, f)$ $(k = 1, 2)$. But since $\sigma(G_0)$ maps the subspace, spanned by j_1 and j_2, of J into itself, and since $(j_k, f) = 0$ $(k = 1, 2)$ (cf. the begin of 4.8), we conclude, that $(j_k, \rho(\gamma^{-1})\, p) = t_k(c, f)$. By virtue of $(c, f) \neq 0$ this implies, that ϕ is a bijection. The same argument shows, that if $l' = \rho(\gamma^{-1})\, p$ varies continuously on O, then so does (γ, p) on $\Gamma \times O$, which suffices to prove our statement. Let us write $\rho(a^{-1})\, \phi(q) = \phi(q\bar{a})$ $(a \in G)$. One verifies easily, that if $q = (\gamma, p)$, then $q\bar{a} = (\gamma\bar{a}, \rho([g_0(\gamma a)]^{-1})\, p)$. We denote the Borel measure $d\gamma dp/\psi(\gamma)$ on O by dq. We have

$$\begin{aligned} d(q\bar{a}) = d(\gamma\bar{a})\, d(\rho([g_0(\gamma a)]^{-1})\, p)/\psi(\gamma\bar{a}) &= (\psi(a)/\psi(g_0(\gamma a) \cdot \gamma\bar{a}))\, d\gamma\, dp \\ &= d\gamma\, dp/\psi(\gamma) = dq; \end{aligned}$$

in other words, dq is invariant under the map $q \to q\bar{a}$ for any fixed a in G. Therefore, the direct image dv of dq under ϕ satisfies all the requirements stated above.

We shall show in 5.5. below, that if $h(l')$ is a rapidly decreasing function on $\mathscr{L}'$, then we have

$$\int_0 |\, h(l')|\, dv < +\infty.$$

Assuming this for the moment, we can write the last equation of e) as

$$\mathrm{Tr}(T(\varphi)) = \int_O \hat{\omega}(l')\, dv,$$

which is what we set out to establish.

5.4. We proceed now to the discussion of the case 4.5, where we shall assume, that $G_1 \neq G_0$.

a) Let J be an ideal as in 4.5. We have a base $\{j_1, j_2\}$ in J, such that $\mathrm{ad}l j_1 = \alpha_1(l) j_1 + \alpha_2(l) j_2$, $\mathrm{ad}l j_2 = -\alpha_2(l) j_1 + \alpha_2(l) j_2$ $(l \in \mathscr{L})$ and $\dim\{\alpha_1, \alpha_2\} = 2$. By virtue of the assumption made above, there exist a positive integer k_0, such that

$$G_1 = \{g; A_2(g) = 2\pi k_0 k, k = 0, \pm 1, \pm 2,...\}$$

$(dA_2 = \alpha_2$; cf. 4.5. f)). We can find elements l_1 and l_2 in $\mathscr{L}$, such that $[l_1, l_2] = 0$, $\alpha_1(l_1) = 1$, $\alpha_2(l_1) = 0$ and $\alpha_1(l_2) = 0$, $\alpha_2(l_2) = k_0$. We put $g_j(t) = \exp(tl_j)$ $(j = 1, 2)$, denote the subalgebra, spanned by l_1 and l_2 by $\mathscr{L}_1$ and set $\Gamma = \exp \mathscr{L}_1$. Any element of G can be written as $g_1\gamma$ $(g_1 \in G_1, \gamma \in \Gamma)$, and $g_1\gamma = g_1'\gamma'$ holds if and only if there is an element δ in $\Delta = \{g_2(2\pi k); k = 0, \pm 1, \pm 2,...\}$ such that $g_1' = g_1\delta^{-1}$, $\gamma' = \delta\gamma$. Given γ in Γ and a in G, we shall denote by $g_1(\gamma a)$ and $\gamma\bar{a}$ any pair of elements, satisfying $\gamma a = g_1(\gamma a)\gamma\bar{a}$, from G_1 and Γ respectively. We shall identify $H = \Gamma/\Delta = G/G_1$ with the subset $I = \{g_1(t_1) g_2(t_2); t_1 \in R, 0 \leqslant t_2 < 2\pi\}$ of Γ. Let $d\gamma$ be a Haar measure on Γ, and dh the corresponding measure on H; we have evidently $d\gamma\bar{a} = d\gamma$.

b) With these notations the representation $T = \mathrm{ind}_{G_1 \uparrow G} V$ (cf. 4.5.g)) can be realized in the Hilbert space, corresponding to the family of functions defined on Γ, taking their values in the space of V, satisfying $f(\gamma\delta) = V(\delta)f(\gamma)$ for any γ in Γ and δ in Δ, and $\int_I \|f(\gamma)\|^2 \, d\gamma < +\infty$. The operator $T(a)$ $(a \in G)$ arises from the map $f(\gamma) \to V(g_1(\gamma a)) f(\gamma\bar{a})$ (cf. 5.3. b)).

c) Let φ be an element in $P(T)$. By computations, as in 5.3. c) we show, that if $f(\gamma)$ is sufficiently regular, we have

$$(T(\varphi)f)(\gamma) = \int_I K_\varphi(\gamma, \gamma') f(\gamma')\, d\gamma' \qquad (\gamma \in I)$$

where

$$K_\varphi(\gamma, \gamma') = (1/\Delta(\gamma)) \int_G \varphi(\gamma^{-1} g_1 \gamma')\, V(g_1)\, dg_1$$

and the right invariant measure dg, used in forming $T(\varphi)$, on G and the right invariant measure dg_1 on G_1 are connected by $dg = dg_1 \,.\, dh$. We conclude also, as before, that $0 \leqslant K_\varphi(\gamma, \gamma) = (1/\Delta(\gamma))\, V(\varphi_\gamma)$, where we have put $\varphi_\gamma(g_1) = \varphi(\gamma^{-1} g_1 \gamma)$ $(g_1 \in G_1)$.

Let us show, that for any fixed γ the support of φ_γ is contained in

D_0. To this end, by virtue of what we saw in 5.3. c) it is enough to establish, that $D \cap G_1 \subset G_0$. But this is clear, since if g belongs to $G_1 - G_0$, we have $A_2(g) = 2\pi m$ ($m \neq 0$), thus g cannot be in D. We recall (cf. 4.5. g)), that the restriction U, of V to G_0, is an irreducible representation, belonging to the orbit O_0 (cf. 4.2), of G_0. We have $0 \leqslant V(\varphi_\gamma) = U(\varphi_\gamma)$, and therefore $\varphi_\gamma \in P(U)$. Since $G_0 = (a)$ and $\dim G_0 + 2 = \dim G$ (cf. 4.5. a)), by virtue of the assumption of our inductive procedure

$$\mathrm{Tr}(U(\varphi_\gamma)) = \int_{O_0} \hat{\omega}_\gamma(l_0')\, dv_0$$

where $\omega_\gamma(l_0) = \mu_0(l_0)\, \varphi(\sigma(\gamma^{-1})\, l_0)$ $(l_0 \in \mathscr{L}_0)$. Taking into account, that the restriction of Δ to the invariant subgroup G_0 coincides with Δ_0, one sees easily, that $g = g_0\gamma$ ($g_0 \in G_0$, $\gamma \in \Gamma$) implies

$$(1/\Delta(\gamma))\, \mathrm{Tr}(U(\varphi_\gamma)) = (1/\Delta(g))\, \mathrm{Tr}(U(\varphi_g)).$$

In this fashion we conclude, as in 5.3. c), that

$$\mathrm{Tr}(T(\varphi)) = \int_I \mathrm{Tr}(K_\varphi(\gamma, \gamma))\, d\gamma = \int_H (1/\Delta(g)) \Big(\int_{O_0} \hat{\omega}_g(l_0')\, dv_0 \Big)\, dh.$$

d) Using $\Delta \mid G_0 \equiv \Delta_0$ and the reasonings of 5.3. d), we show, that there exist a function $\mu(l)$ of the form as in 5.1, such that $\mu \mid \mathscr{L}_0 \equiv \mu_0$, and therefore $\omega_g(l_0) \equiv \omega(\sigma(g^{-1})\, l_0)$ $(l_0 \in \mathscr{L}_0)$, where $\omega(l) \equiv \mu(l)\, \varphi(l)$ $(l \in \mathscr{L})$. From this, proceeding as in 5.3. e) we deduce, that

$$\mathrm{Tr}(T(\varphi)) = \int_H \Big(\int_{O_0} \Big(\int_{\mathscr{L}_0^\perp} \hat{\omega}(\rho(g^{-1})(l' + l_0^\perp))\, dl_0^\perp \Big)\, dv_0 \Big)\, dh.$$

e) Let us show, that there is a positive invariant Borel measure dv on O, such that

$$\int_O k(l')\, dv = \int_H \Big(\int_{O_0} \Big(\int_{\mathscr{L}_0^\perp} k(\rho(g^{-1})(l' + l_0^\perp))\, dl_0^\perp \Big)\, dv_0 \Big)\, dh$$

for any function $k(l')$, which is continuous and of a compact support on O. To this end we observe first, that dv_0 is invariant under the action of $\rho(G_1) \mid \mathscr{L}'/\mathscr{L}_0^\perp$. In fact, let τ be any transformation of this kind. O_0 carries a 2-form ω invariant under $\rho_0(G_0)$ (cf. 5.6 below), and dv_0 is a constant multiple of $(\omega)^{d/2}$ $(d = \dim O_0)$. Therefore it is enough to verify, that ω is invariant under τ, too, which is trivial.

From this we conclude, that there exist a positive Borel measure dp, invariant under $\rho(G_1)$, on $O_1 = \rho(G_0)f$, such that

$$\int_{O_1} k(p)\,dp = \int_{O_0} \Big(\int_{\mathscr{L}_0^\perp} k(l' + l_0^\perp)\,dl_0^\perp\Big)\,dv_0 .$$

This implies, that the continuous function $K(g) = \int_{O_1} k(\rho(g^{-1})\,p)\,dp$ on G satisfies $K(g_1 g) = K(g)$ $(g_1 \in G_1,\ g \in G)$, and therefore K can be viewed as a continuous function on $H = G/G_1$. Let us show, that its support is compact. To this end we note, that $\rho(g^{-1})\,p = \rho(g'^{-1})\,p'$ $(g, g' \in G, p, p' \in O_1)$ yields $g' = g_1 g$ $(g_1 \in G)$. Therefore, since $A_1 \mid G_1 \equiv 1$ (cf. 4.5. f)), there is a well defined function ϕ on O, such that $\phi(l') = A_1(g)$ if $l' = \rho(g^{-1})\,p$, and to prove our statement it suffices to show, that ϕ is continuous. But this follows at once from the relation

$$\exp(2A_1(g))[(j_1, f)^2 + (j_2, f)^2] = (j_1, \rho(g^{-1})\,p)^2 + (j_2, \rho(g^{-1})\,p)^2.$$

In this fashion we conclude, that there is a positive Borel measure dv on O, such that

$$\int_O k(l')\,dv = \int_H \Big(\int_{O_0} \Big(\int_{\mathscr{L}_0^\perp} k(\rho(g^{-1})(l' + l_0^\perp))\,dl_0^\perp\Big)\,dv_0\Big)\,dh.$$

Its invariance under ρ is implied by the fact, that dh is an invariant measure on H. We shall show below in 5.5, that if $k(l')$ is rapidly on $\mathscr{L}'$, then

$$\int_O |\,k(l')|\,dv < +\infty.$$

On the basis of this, the final formula of d) yields

$$\mathrm{Tr}(T(\varphi)) = \int_O \hat{\omega}(l')\,dv.$$

5.5. Our purpose here is to give a proof for the convergence relation just quoted.

a) First we observe, that the proof of Lemma 8 in [*17*] (cf. Part 5) yields the following result. Let $\mathscr{L}$ be a solvable Lie algebra, $\mathfrak{n}$ its nilradical, and $\mathscr{L}_1$ a subalgebra containing $\mathfrak{n}$. We assume, that, for any l in $\mathscr{L}_1$, the roots of ad l are real, and that ad $\mathscr{L}_1$ is algebraic in

$L(\mathscr{L}_1)$. Let us put $G_1 = \exp \mathscr{L}_1$, and for a fixed f in $\mathscr{L}'$ let us form the orbit $O_1 = \rho(G_1)f$, which we suppose to be closed. Then O_1 carries a positive invariant Borel measure dv_1, and we have for any rapidly decreasing function $h(l')$ on $\mathscr{L}'$

$$\int_{O_1} |h(l')|\, dv_1 < +\infty.$$

b) Let $\mathfrak{a}$ be an abelian algebraic Lie algebra of endomorphisms acting on a finite dimensional real vector space V. We denote by $\{\lambda_j;\ 1 \leqslant j \leqslant M\}$ the collection of roots of $\mathfrak{a}$. We know (cf. 2.1. a)), that $\mathfrak{a}_C$ contains a base $\{A_k;\ 1 \leqslant k \leqslant K\}$, such that the numbers $\lambda_j(A_k)$ $(1 \leqslant j \leqslant M,\ 1 \leqslant k \leqslant K)$ are integers. From this we deduce easily the following conclusion: $\mathfrak{a}$ possesses a base $\{a_k, b_j; 1 \leqslant k \leqslant R,\ 1 \leqslant j \leqslant S\}$, such that the numbers $\lambda_r(a_k)$, $\lambda_s(b_j)$ are integers and $\sqrt{-1}$ times integers respectively. Let us denote by $\mathfrak{a}_r$ and $\mathfrak{a}_i$ the subspaces, spanned by the systems $\{a_k\}$ and $\{b_j\}$ respectively, of $\mathfrak{a}$. $\mathfrak{a}_r$ and $\mathfrak{a}_i$ are abelian algebraic Lie algebras of semi-simple endomorphisms of V, of which $\mathfrak{a}$ is the direct sum. The eigenvalues of $a \in \mathfrak{a}$ are real or purely imaginary, if it belongs to $\mathfrak{a}_r$ or $\mathfrak{a}_i$ respectively.

c) Let us assume now, that $\mathscr{L}$ is as in 5.1, and let us consider the decomposition $\mathscr{L} = \mathfrak{n} + \mathfrak{a}$ as in 2.1. b). The restriction of the adjoint representation of $\mathscr{L}$ to $\mathfrak{a}$ is an isomorphism with its image, which is an abelian algebraic Lie algebra of semi-simple endomorphisms of ad $\mathscr{L}$. We apply now b) to ad $\mathfrak{a}$ (in place of $\mathfrak{a}$), and define $\mathfrak{a}_r$ and $\mathfrak{a}_i$ by $\mathrm{ad}(\mathfrak{a}_r) = (\mathrm{ad}\ \mathfrak{a})_r$ and $\mathrm{ad}(\mathfrak{a}_i) = (\mathrm{ad}\ \mathfrak{a})_i$. Let O be the closed orbit of 5.1, and f an element in O. We write $\mathscr{L}_1 = \mathfrak{n} + \mathfrak{a}_r$ and show, that $\mathscr{L}_1$ satisfies the conditions, with respect to f as just chosen, of a) above. To this end, it is enough to establish, that $O_1 = \rho(G_1)f$ is closed in $\mathscr{L}'$. Let us denote by R the Lie algebra of the stable group of f in G; we verify easily by recalling, that $R = \{r; r \in \mathscr{L}, (\mathrm{ad}(r))'f = 0\}$, that ad R is an algebraic subalgebra of $L(\mathscr{L})$. Setting $\mathscr{L}_2 = \mathscr{L}_1 + R$ we get ad $\mathscr{L}_2 =$ ad $\mathscr{L}_1 +$ ad R, which implies (cf. the end of 2.1. a) and [5], Theorem 14, p. 175), that ad $\mathscr{L}_2$, too, is algebraic. If $G_2 = \exp \mathscr{L}_2$, we have evidently $0_1 = \rho(G_1)f = \rho(G_2)f$, and $\dim O - \dim O_1 = \dim G - \dim G_2$, and thus the desired conclusion follows from the Remark of 4.3. f).

d) We put $A = \exp \mathfrak{a}_i$ and $\Lambda = \rho(A)$; Λ is a compact abelian subgroup, isomorphic to T^S, of $GL(\mathscr{L}')$. Since $G = AG_1$, we have $O = U_{\lambda \in \Lambda} \lambda O_1$. If $h(l')$ is continuous and of a compact support on $\mathscr{L}'$, then the function $h(\lambda p)$ $(\lambda \in \Lambda,\ p \in O_1)$ has the same property on

$\Lambda \times O_1$, and one sees easily (cf. [17], Lemma 8, II), that there is an invariant measure $d\lambda$ on Λ, such that for any h of the indicated sort

$$\int_O h(l')\,dv = \int_{\Lambda\times O_1} h(\lambda p)\,dv_1\,d\lambda.$$

e) Let us assume finally, that $h(l')$ is rapidly decreasing on $\mathscr{L}'$. Then $H(l') = \int_\Lambda |\,h(\lambda l')|\,d\lambda$ $(l' \in \mathscr{L}')$, too, is rapidly decreasing, and therefore, by virtue of a) and c)

$$+\infty > \int_O H(p)\,dv = \int_{\Lambda\times O_1} |\,h(\lambda p)|\,d\lambda\,dv = \int_O |\,h(l')|\,dv,$$

which is what we wished to establish.

5.6. We proceed now to discuss an algorithm to compute the canonical measure dv (cf. 5.1), which generalizes the situation found in the nilpotent and exponential case (cf. [16] and [17], Proposition 4). Let for a moment G be an arbitrary connected Lie group with the Lie algebra $\mathscr{L}$. If O is an orbit of positive dimension of the coadjoint representation ρ, acting on $\mathscr{L}'$, of G, we can assign to it an invariant measure as follows. Fixing an arbitrary element p of O, let us consider the map α_p from G onto O defined by $\alpha_p(a) = \rho(a)\,p$ $(a \in G)$. Its differential $\varphi_p = d\alpha_p$ is a map of $\mathscr{L}$ onto the tangent space T_p of O at p, and its kernel, which is identical with the Lie algebra of the stabilizer of p, coincides with the radical of the skew-symmetric bilinear form $B_p(l_1, l_2) = ([l_1, l_2], p)$ on $\mathscr{L} \times \mathscr{L}$. In this fashion we can conclude, that there exist a well determined nondegenerate skew-symmetric bilinear form ω_p on $T_p \times T_p$, such that $\delta\alpha_p(\omega_p) = B_p$. Varying p on O we obtain a 2-form, which can easily be seen invariant under the action of G (cf. [16], p. 256). Let d be the dimension of O; by what preceeds, this is necessarily an even number. The exterior power $(\omega)^{d/2}$ is an invariant differential form of maximal rank; we denote by dw the corresponding positive invariant measure, and call it the Kostant measure of O.

From now on we assume again, that $G = (a)$ (cf. 2.1. c)). It is clear from the beginning, that the canonical measure, being a positive invariant measure on O, is uniquely determined up to a positive multiplicative constant. The formula (cf. 5.1)

$$\mathrm{Tr}(T(\varphi)) = \int_O \hat{\omega}(l')\,dv$$

shows that it is completely determined, provided we have made a choice of the right invariant Haar measure da and of the translation invariant measure dl on $\mathscr{L}$, used in forming the left and right hand side respectively. If we change the normalization of these two measures. then, for any closed orbit O, dv gets multiplied by a constant, independent of the particular choice of O. Consequently, the canonical measure is uniquely determined on each orbit if we assume, as we shall in the sequel, that the ratio of the inverse image, under the canonical map, of da and of dl at the neutral element of $\mathscr{L}$ (denoted by $da/dl\,|_0$) equals one.

It was conjectured by B. Kostant, that, using the previous notations, $dv = dw/C(d)$, where $C(d) = (d/2)!\pi^{d/2} \,.\, 2^d$. We proved the validity of this formula for the nilpotent and the exponential case in previous papers (cf. [*16*] and [*17*], loc. cit.), and shall now extend it to the class of groups considered in this paper. To this end we adopt the inductive procedure started in 5.1. Let us observe, that for the step discussed in 5.2 the desired conclusion is immediately clear. In the following we shall consider in detail only the case of 5.3, assuming dim $O > 4$, and leave the easy modifications necessary to settle the remaining cases to the reader. The proof, along with that of the Theorem of 5.1, will be completed in Section 6 (cf. 6.7).

In what follows we shall use the notations of 4.8 and 5.3 without further explanation.

a) Let us start by observing, that if $\{k_r;\ 1 \leqslant r \leqslant d\}$ is a supplementary base to R in $\mathscr{L}$, then, writing $g_j(t) = \exp(tk_j)$, $T = (t_1, t_2, \ldots, t_d) \in R^d$ and $g(T) = g_1(t_1)\, g_2(t_2) \cdots g_d(t_d)$, through $T \to \rho([g(T)]^{-1})f$ the variables of T define a system of local coordinates, valid around f, on O. For later purposes, we fix the base k_j in the following fashion. We set first $k_1 = j_1$ and $k_2 = j_2$. We assume next, that $\{k_j;\ 3 \leqslant j \leqslant d-2\}$ is a supplementary base to R_0 in $\mathscr{L}_0$, such that, for $r \leqslant s$, $B(k_r, k_s) = 1$ if $r = 2i-1$, $s = 2i$, and 0 otherwise ($B(l_1, l_2) = ([l_1, l_2], f)$ for $l_1, l_2 \in \mathscr{L}$). We have, of course, $B(k_r, k_s) = 0$ ($r = 1, 2;\ 3 \leqslant s \leqslant d-2$). Finally, since $[j_1, j_2] = 0$, we can find elements k_{d-1} and k_d such that $B(k_1, k_{d-2+j}) = \delta_{ij}$ $(i, j = 1, 2)$, and $B(k_r, k_{d-2+s}) = 0$ ($3 \leqslant r \leqslant d-2$, $s = 1, 2$). In particular, we have in this fashion at f

$$\omega = 2\left(dt_1 \wedge dt_{d-1} + dt_2 \wedge dt_d + \sum_{s=2}^{(d-2)/2} dt_{2s-1} \wedge dt_{2s}\right)$$

and therefore, putting $m = d/2$

$$\omega^m = m!\,.\,2^m \prod_{j=1}^{d} \wedge\, dt_j$$

and thus, at f, $dw/C(d) = (2\pi)^{-m} dt_1\, dt_2 \cdots dt_d$.

Let us write $d_0 = d - 4$, $m_0 = d_0/2$, $T_0 = (t_3, t_4, \ldots, t_{d-2}) \in R^{d_0}$ and $g_0(T_0) = g_3(t_3) \cdots g_{d-2}(t_{d-2})$, and in general, let us distinguish notions, relative to G_0, by an index zero. We have, as above, that through $T_0 \to g_0(T_0)\,\pi(f)$, the variables of T_0 determine a system of local coordinates, valid around $\pi(f)$, on O_0, and hence at $\pi(f) : dw_0/C(d_0) = (2\pi)^{-m_0}\, dt_3 \cdots dt_{d-2}$. By virtue of the assumption of our induction we have $dv_0 = dw_0/C(d_0)$, and thus it will be enough to show, that at f, $dv = (2\pi)^{-2}\, dt_1\, dt_2\, dv_0\, dt_{d-1}\, dt_d$.

b) We recall (cf. 5.3. f)), that we have

$$\int_O h(l')\, dv = \int_\Gamma \left(\int_{O_0} \left(\int_{\mathscr{L}_0^\perp} h(\rho(\gamma^{-1})(l' + l_0^\perp))\, dl_0^\perp \right) dv_0 \right) d\gamma/\psi(\gamma).$$

In the following we shall assume, that the support of the smooth function h is contained in a neighborhood of f, where the coordinates $\{t_r\}$ are valid. Let us also recall, that the measure $d\gamma$ on Γ, which is identifiable to the homogeneous space G/G_0, is determined by the condition $\psi(g)\, dg = dg_0\, d\gamma$ (cf. 5.3. a)). To obtain $dl_0^\perp$, we take linear measures dl and dl_0 on $\mathscr{L}$ and $\mathscr{L}_0$ respectively, such that $dg/dl\,|_0 = 1$ and $dg_0/dl_0\,|_0 = 1$, form their duals dl' and dl_0', and choose the linear measure $dl_0^\perp$ on $\mathscr{L}_0^\perp$ such that $dl' = dl_0'\, dl_0^\perp$ hold (cf. 5.3. e)).

c) We have, since $\det(\rho(a) \mid \mathscr{L}_0^\perp) \equiv \psi(a)$ $(a \in G_0)$, for each fixed γ

$$\int_{O_0} \left(\int_{\mathscr{L}_0^\perp} h(\rho(\gamma^{-1})(l' + l_0'))\, dl_0^\perp \right) dv_0$$

$$= \int_{O_0} \left(\int_{\mathscr{L}_0^\perp} h(\rho([g_0(T_0)\,\gamma]^{-1})(f + l_0^\perp))\, dl_0^\perp \right) \psi(g_0(T_0))\, dv_0 .$$

We assume, as we obviously can, that

$$dl = dl_0 dy_1 dy_2 \qquad (l = l_0 + y_1 k_d + y_2 k_d).$$

We have $B(k_i, k_{d-2+r}) = -(\operatorname{ad} k_{d-2+r} j_i, f) = -\lambda_i(k_{d-2+r}) = \delta_{ir}$ $(i, r = 1, 2)$, and therefore $dl_0^\perp = (2\pi)^{-2}\, dy_1'\, dy_2'$ $(l_0^\perp = y_1'\lambda_1 + y_2'\lambda_2)$. On the other hand (cf. 4.8. b) and the end of 4.9 for the general case) $(\exp(t_1 j_1) \exp t_2 j_2))\, f = f + t_1\lambda_1 + t_2\lambda_2$ and consequently

$$\int_{O_0} \left(\int_{\mathscr{L}_0^\perp} h(\rho(\gamma^{-1})(l' + l_0^\perp))\, dl_0^\perp \right) dv_0$$

$$= (2\pi)^{-2} \int_{O_0} \left(\int_{\mathscr{L}_0^\perp} h(\rho([g_1\gamma]^{-1})\, f)\, dt_1\, dt_2 \right) \psi(g_1)\, dv_0$$

where $g_1 = g_1(t_1)\, g_2(t_2)\, g_0(T_0)$.

d) We can take in 5.3. a) $l_i = -k_{d-2+i}$ $(i = 1, 2)$; by virtue of $dg/dl\,|_0 = 1$ and $dg_0/dl_0\,|_0 = 1$ we have $dg = dg_0\, dt_{d-1}\, dt_d$ at the neutral element of G. Therefore, finally

$$\int_O h(l')\, dv = (2\pi)^{-2} \int_\Gamma \left(\int_{O_0} \left(\int_{\mathscr{L}_0^{\perp}} h([g(T)]^{-1} f)\, a(T)\, dt_1\, dt_2 \right) dv_0 \right) dt_{d-1}\, dt_d$$

where $a(0) = 1$, and thus at $f : dv = (2\pi)^{-2}\, dt_1\, dt_2\, dv_0\, dt_{d-1}\, dt_d$, which is the desired conclusion (cf. the end of a)).

6. The purpose of this concluding section is to complete the proof of the Theorem of 5.1. In order to do this, by virtue of 4.10 and the results of Section 5 it suffices to verify our main formula (cf. 5.1) under the following assumptions : $\mathscr{L} = \mathfrak{n} + \mathfrak{a}$, where the nilradical $\mathfrak{n}$ is spanned by $\{p_i, q_k, c;\ i, k = 1, 2, \ldots, n\}$ satisfying $[p_i, q_k] = \delta_{ik} c$, all other brackets being zero; the roots of ad a, for any element a of the abelian algebra $\mathfrak{a}$, such that ad $\mathfrak{a}$ is algebraic and consists of semi-simple endomorphisms, are purely imaginary; finally, the irreducible representation T belongs to an orbit 0, for which $(c, f) \neq 0$ $(f \in 0)$. In this case evidently $\Delta \equiv 1$, and we shall see that, as already mentioned in Section 1, in the expression of $\mu(l)$ $\mathscr{F}_1$ is empty.

Let us put $\mathrm{ad}\, l p_k = \varphi_k(l)\, q_k$ and $\mathrm{ad}\, l q_k = -\varphi_k(l)\, p_k$ $(l \in \mathscr{L}$, $k = 1, 2, \ldots, n)$; the linear forms $\{\pm i\varphi_k\}$ are the roots of $\mathscr{L}$, and therefore vanish on $\mathfrak{n}$. We know (cf. 5.5. b)), that $\mathfrak{a}$ contains a base $\{a_k;\ k = 1, 2, \ldots, m\}$, such that if $a = \sum_{k=1}^m t_k a_k$, $T = (t_1, t_2, \ldots, t_m)$ and $\varphi_k(a) = \varphi_k(T)$, the $\varphi_k(T)$'s are linear forms with integral coefficients. Since $\mathrm{ad}(a) = 0$ $(a \in \mathfrak{a})$ implies $a = 0$, we have $m \leqslant n$, and we can assume, that the system $\{\varphi_1, \varphi_2, \ldots, \varphi_m\}$ is linearly independent, and that even $\varphi_k(T) = b_k t_k$ $(1 \leqslant k \leqslant m)$ and $\varphi_k(T) = \sum_{j=1}^m b_{kj} t_j$ $(k = m + 1, \ldots, n)$, where the coefficients are integers. We shall write in the following $p_i = e_{2i-1}$, $q_i = e_{2i}$ $(i = 1, 2, \ldots, n)$ and $c = e_{2n+1}$.

6.1. Let us denote by $\{e_i', a_k'\}$ a dual base in $\mathscr{L}'$, and let us write (τ, y) $(\tau = (\tau_1, \tau_2, \ldots, \tau_m)$, $y = (y_1, y_2, \ldots, y_{2n}))$ for the element $l' = \sum_{k=1}^m \tau_k a_k' + \sum_{i=1}^{2n+1} y_i e_i'$. Let O be an orbit, which is not orthogonal to the center C of $\mathscr{L}$. We start by showing, that

$$O = \Big\{ (\tau, y);\ \tau_i = \lambda_i - (b_i/2\gamma)(y_{2i-1}^2 + y_{2i}^2) - (1/2\gamma)\Big(\sum_{j=m+1}^{n} b_{ji}(y_{2j-1}^2 + y_{2j}^2) \Big);$$
$$y_i = \text{arbitrary for } 1 \leqslant i \leqslant 2n,\ y_{2n+1} = \gamma \Big\};$$

here λ_i $(1 \leqslant i \leqslant m)$ and $\gamma \neq 0$ are constants, uniquely characterizing O (observe, that $\gamma = (e_{2n+1}, f), f \in O)$.

a) Let us consider first an element g in $\mathfrak{n}'$ such that $\gamma = (e_{2n+1}, g) \neq 0$, and let us denote by O_N the orbit, with respect to the coadjoint representation ρ_0 of $N = \exp \mathfrak{n}$, containing g. Writing E for the orthogonal complement, in $\mathfrak{n}'$, of $\{e_{2n+1}\}$ we claim, that $O_N = g + E$. In fact, one verifies easily, that the radical of the bilinear form $([x, y], g)$ on $\mathfrak{n} \times \mathfrak{n}$ is $\{e_{2n+1}\}$, and consequently $\dim O_N = 2n$. On the other hand, we have $(e_{2n+1}, \rho_0(a) g) \equiv (e_{2n+1}, g)$ for all a in N, and thus O_N is contained and open in $g + E$. But O_N, being the orbit of the unipotent representation ρ_0, is necessarily closed, and therefore $O_N = g + E$.

We denote by π the canonical projection from $\mathscr{L}'$ onto $\mathfrak{n}' = \mathscr{L}'/\mathfrak{n}^\perp$, and assume $g = \pi(f)$, where f is some element in O. Since $\rho_0 = \pi \circ (\rho \mid N)$, the above assertion yields $O = O_N$.

b) Let us consider the system of polynomial functions

$$F_0(\tau, y) \equiv y_{2n+1} \text{ and } F_i(\tau, y) \equiv 2\tau_i y_{2n+1} + b_i(y_{2i-1}^2 + y_{2i}^2) + \sum_{j=m+1}^{n} b_{ji}(y_{2j-1}^2 + y_{2j}^2)$$

$(1 \leqslant i \leqslant m)$ on $\mathscr{L}'$. An easy computation, the details of which we omit, shows, that these are annihilated by the derivations, continuing the elements of $(\operatorname{ad} \mathscr{L})'$, and hence are invariants of ρ. We denote by $2\lambda_i\gamma$ the value of F_i $(1 \leqslant i \leqslant m)$ on O; thus we have for any $(\tau, y) \in O$: $\tau_i = \lambda_i - (b_i/2\gamma)(y_{2i-1}^2 + y_{2i}^2) - (1/2\gamma)(\sum_{j=m+1}^{n} b_{ji}(y_{2j-1}^2 + y_{2j}^2))$. But since by a) y_k $(1 \leqslant k \leqslant n)$ can take arbitrary values, O is as claimed above.

6.2. We proceed now to the construction of the irreducible representation T belonging to O (as above). We observe first, that since O is simply connected, T is uniquely determined up to a unitary equivalence (cf. 3.5). Furthermore f in O can always be chosen such that $f = \sum_{k=1}^{m} \lambda_k a_k' + \gamma e_{2n+1}'$ $(\gamma \neq 0)$ and then is uniquely determined.

Let us consider the complex subspace $\not{f}$, spanned by $\{a_k; e_{2j-1} + ie_{2j}$ $(1 \leqslant j \leqslant n)$; $e_{2n+1}\}$, of $\mathscr{L}_C$. It is a subalgebra, and we are going to show, that if $\gamma > 0$, then $\not{f} = (A)$ with respect to f. To this end, we have to verify, that $\not{f}$ satisfies all the conditions of 3.1. We start by observing, that $[\not{f}, \not{f}]$ is contained in the complex subspace, spanned by $\{e_{2j-1} + ie_{2j};\ 1 \leqslant j \leqslant n\}$, of $\mathscr{L}_C$, and hence is orthogonal to f. Therefore, to prove, that $\not{f}$ is maximal self orthogonal, it is enough to show, that $n + m + 1 = \dim \mathfrak{f} = \frac{1}{2}(\dim \mathscr{L} + \dim R)$. But this follows at once from the easily verifiable fact, that R is the subspace, spanned by the system of $m + 1$ elements $\{e_{2n+1}, a_k;\ 1 \leqslant k \leqslant m\}$, of $\mathscr{L}$. Condition II loc. cit. is trivially fulfilled, since, O being simply connected, S is connected and hence is contained in $H = \exp \mathfrak{f}$. We have $\mathfrak{f} + \bar{\mathfrak{f}} = \mathscr{L}_C$, whence III α). To check III β), we can assume, that $x + iy$, in $\mathfrak{f}$, is of the form $\sum_{j=1}^{n} (v_{2j-1} + iv_{2j})(e_{2j-1} + ie_{2j})$. But then $x = \sum_{j=1}^{n} (v_{2j-1}e_{2j-1} - v_{2j}e_{2j})$, $y = \sum_{j=1}^{n} (a_{2j}e_{2j-1} + a_{2j-1}e_{2j})$, and thus $([x, y], f) = \gamma(\sum_{j=1}^{n} (v_{2j-1}^2 + v_{2j}^2)) \geqslant 0$ and $([x, y], f) = 0$ implies, that x and y belong to $R = \mathfrak{f} \cap \mathscr{L}$. $\mathfrak{f} \cap \mathfrak{n}_C$ is spanned by $\{e_{2j-1} + ie_{2j};\ 1 \leqslant j \leqslant n;\ e_{2n+1}\}$, and therefore we have $\dim(\mathfrak{f} \cap \mathfrak{n}_C) = n + 1$; but at the same time also $\dim \mathfrak{n} - \frac{1}{2}\dim 0_N = (2n + 1) - n = n + 1$ (cf. 6.1. a)), proving our point.

One verifies easily, that if $\gamma < 0$, the subalgebra $\bar{\mathfrak{f}}$, of $\mathscr{L}_C$, satisfies $\mathfrak{f} = (A)$.

Let us suppose again $\gamma = (e_{2n+1}, f) > 0$. By what we saw above, we have $d = \mathfrak{f} \cap \mathscr{L} = R$, $D = S = A$ and evidently $\Delta_A \equiv 1$, $\Delta_G \equiv 1$ (cf. 3.2). To obtain a realization of $G = \exp \mathscr{L}$, we consider the collection of all systems $(t_1, t_2, \ldots, t_m;\ v_1, v_2, \ldots, v_n;\ u)$, where u, t_k $(1 \leqslant k \leqslant m)$ are arbitrary real, and v_j $(1 \leqslant j \leqslant n)$ arbitrary complex numbers, and define multiplication by

$$(t_1, \ldots, t_m;\ v_1, \ldots, v_n;\ u)(t_1', \ldots, t_m';\ v_1', \ldots, v_n';\ u')$$

$$= \Big(t_1 + t_1', \ldots, t_m + t_m';\ v_1 + e^{i\varphi_1(t)}v_1', \ldots, v_n + e^{i\varphi_m(t)}v';$$

$$u + u' + \tfrac{1}{2}\Big(\sum_{j=1}^{n} \operatorname{Im}(\bar{v}_j v_j' e^{i\varphi_j(t)})\Big)\Big).$$

We have $A = \{a;\ a \in G,\ a = (t_1, \ldots, t_m;\ 0, \ldots, 0;\ u)\}$, and, putting $\lambda t = \sum_{k=1}^{m} \lambda_k t_k$, $\chi(a) = \exp(i\lambda t + i\gamma u)$. Since, as one verifies easily, $H \cap E = H \cap G = D$ ($E = \exp e$, $e = (\mathfrak{f} + \bar{\mathfrak{f}}) \cap \mathscr{L} = \mathscr{L}$), when forming, as in 3.4. d), $H_1(\chi, G)$, we can take $U_E = E$ ($HE = G_C$), and of course $\omega \equiv 1$. Taking into account all this, we obtain the following realization of T (for this cf. also [*10*], Part III, in particular 3.6). The representation space $H(T)$ is the Hilbert space of all functions, holomorphic and entire in the n complex variables $z = (z_1, z_2, \ldots, z_n)$, and satisfying the condition

$$\int_{C^n} |f(z)|^2 \exp(-\tfrac{1}{2}\gamma\ |z|^2)\, dz < +\infty,$$

where $|z| = (\sum_{k=1}^{n} |z_k|^2)^{1/2}$ and $dz = dx_1\, dy_1 \cdots dx_n\, dy_n$ $(z_k = x_k + iy_k)$. If $a = (t_1, \ldots, t_m;\, v_1, \ldots, v_n;\, u)$ and $f \in H(T)$, we have

$$\begin{aligned}(T(a)f)(z) = {} & \exp(i\lambda t)\exp(i\gamma u)\exp(-\tfrac{1}{4}\gamma\, |v|^2) \\ & \times \exp(-\tfrac{1}{2}\gamma z\bar{v}) f(e^{-i\varphi_1(t)}(z_1 + v_1), \ldots, e^{-i\varphi_n(t)}(z_n + v_n)) \\ & \times \left(zv = \sum_{k=1}^{n} z_k \bar{v}_k\right).\end{aligned}$$

Let us write T' for the representation, belonging to the orbit $-O$. A simple verification shows, that T' can be realized in the following fashion. As representation space $H(T')$ we take the collection of all functions, antiholomorphic and entire in $(z_1, z_2, \ldots, z_n)$ with a metric as above. The map $f \to Sf = \bar{f}$ $(f \in H(T'))$ is a conjugate linear isometry from $H(T')$ onto $H(T)$, and we define $T'(a)$ as $S^{-1}T(a)\, S$ $(a \in G)$.

Let us observe, that the law of composition as above defines the structure of a solvable Lie group on $R^m \times C^n \times R'$ even if the coefficients of the linear forms $\{\varphi_k(t)\}$ are not integral, and the construction just given yields irreducible representations of this group. Also, the parametrization, given in 6.1, of the orbits, not orthogonal to the center, remains in force. In the sequel we shall compute the trace of $T(\varphi)$ $(\varphi \in P(T))$ for any such representation; observe, that the corresponding group, in general, is not of type I.

6.3. Let O be an orbit, as in 6.1, with $\gamma = 1$. One shows easily, that $d\bar{y} = dy_1\, dy_2 \cdots dy_{2n}$ defines an invariant measure on O. The following result will be used in 6.5. *Let*

$$\varphi(t, x)\ (t = (t_1, t_2, \ldots, t_m) \in R^m,\ x = (x_1, x_2, \ldots, x_{2n+1}) \in R^{2n+1})$$

be a C_c^∞ function on R^{m+2n+1}. Writing, for, $t, \tau \in R^m$, $t\tau = \sum_{k=1}^{m} t_k\tau_k$ etc., *and $dt = dt_1\, dt_2 \cdots dt_m$, we set*

$$\hat{\varphi}(\tau, y) = \int_{R^{m+2n+1}} \varphi(t, x)\, e^{i(t\tau + xv)}\, dt\, dx.$$

For a $t \in R$, satisfying $\prod_{j=1}^{n} \varphi_j(t) \neq 0$, let us form the function

$$\begin{aligned}\psi(t) = {} & \exp(i\lambda t) \prod_{j=1}^{n} [i\varphi_j(t)]^{-1} \\ & \times \int_{R^{2n+1}} \exp\left(i\left(\sum_{j=1}^{n} (x_{2j-1}^2 + x_{2j}^2)/2\varphi_j(t)\right)\right) \varphi(t, x) \exp(ix_{2n+1})\, dx.\end{aligned}$$

Then $|\psi(t)|$ *is bounded and of a compact support, and we have*

$$\int_O \hat{\varphi}(\tau, y)\, dy = (2\pi)^n \int_{R^m} \psi(t)\, dt.$$

a) To prove this assertion, let us set for $T > 0 : C_T = \{\bar{y};\ \bar{y} = (y_1, y_2, \ldots, y_{2n}) \in R^{2n}, |y_k| < T \text{ for all } k\}$. Then we have

$$\int_O \hat{\varphi}(\tau, y)\, d\bar{y} = \lim_{T\to+\infty} \int_{R^{m+2n+1}} \varphi(t, x) F_T(t, x)\, dt\, dx$$

where

$$F_T(t, x) = \int_{C_T} \exp\left(i\left(\sum_{j=1}^{m} \tau_j(\bar{y})\, t_j\right) + i\left(\sum_{j=1}^{2n} x_j y_j\right) + ix_{2n+1}\right) d\bar{y},$$

and $\tau_j(\bar{y}) = \lambda_j - \frac{1}{2} b_j(y_{2j-1}^2 + y_{2j}^2) - \frac{1}{2}(\sum_{k=m+1}^{n} b_{kj}(y_{2k-1}^2 + y_{2k}^2))$ $(j = 1, 2, \ldots, m)$. We have

$$\sum_{j=1}^{m} \tau_j(\bar{y})\, t_j + \sum_{j=1}^{2n} x_j y_j = \lambda t + \sum_{j=1}^{n} (-\tfrac{1}{2}\varphi_j(t)(y_{2j-1}^2 + y_{2j}^2) + (x_{2j-1}y_{2j-1} + x_{2j}y_{2j})),$$

and therefore, putting for any real x and y

$$G_T(x, y) = \int_{-T}^{T} \exp(-\tfrac{1}{2} iu^2 x + iuy)\, du,$$

and for $\bar{x} = (x_1, x_2, \ldots, x_{2n}) \in R^{2n}$,

$$H_T(t, \bar{x}) = \prod_{j=1}^{n} G_T(\varphi_j(t), x_{2j-1})\, G_T(\varphi_j(t), x_{2j}),$$

we get

$$F_T(t, x) = \exp(i\lambda t + ix_{2n+1})\, H_T(t, \bar{x}).$$

b) We show next, that for any fixed t with $\prod_{j=1}^{n} \varphi_j(t) \neq 0$ we have

$$\lim_{T\to+\infty} H_T(t, \bar{x}) = (2\pi)^n \prod_{j=1}^{n} [i\varphi_j(t)]^{-1} \prod_{j=1}^{n} [\exp(i(x_{2j-1}^2 + x_{2j}^2))/2\varphi_j(t)]$$

and that there exist a positive constant $C(t)$, not depending on T and $\bar{x}$, such that $|H_T(t, x)| < C(t)$. To this end, we observe first, that for any $x \neq 0$

$$\lim_{T\to+\infty} G_T(x, y) = (\pi/|x|)^{1/2}(1 - (\operatorname{sg} x)\, i) \exp(iy^2/2x)$$

and therefore

$$\lim_{T\to+\infty} H_T(t, \bar{x}) = \pi^n \prod_{j=1}^{n} [|\varphi_j(t)|]^{-1} \prod_{j=1}^{n} (1 - i\,\mathrm{sg}(\varphi_j(t)))^2$$
$$\times \prod_{j=1}^{n} [\exp(i(x_{2j-1}^2 + x_{2j}^2)/2\varphi_j(t)].$$

In this fashion our first assertion follows from

$$\prod_{j=1}^{n} |\varphi_j(t)|^{-1} \prod_{j=1}^{n} (1 - i\,\mathrm{sg}(\varphi_j(t)))^2 = 2^n \prod_{j=1}^{n} [i\varphi_j(t)]^{-1}.$$

Since there is a positive constant K such that

$$\left| \int_V^U e^{iu^2}\, du \right| < K$$

for all U and V, we conclude easily, that for any fixed $x \neq 0$, we can find a constant $C(x)$, such that $| G_T(x, y)| < C(x)$ for all T and y, implying our second assertion.

c) We have, by virtue of what we have just seen, for any fixed t as above

$$\lim_{T\to+\infty} \int_{R^{2n+1}} \varphi(t, x) F_T(t, x)\, dt\, dx = \psi(t).$$

Let us put $\phi_T(t) = \int_{R^{2n+1}} \varphi(t, x) \exp(ix_{2m+1})\, H_T(t, \bar{x})\, dx$; to complete our proof, it is enough to show, that there is a constant $M > 0$, not depending on T and t, such that we have $|\phi_T(t)| < M$. To this end let us write

$$\varphi_1(t, x) = \int_R \varphi(t, \bar{x}, x_{2n+1}) \exp(ix_{2n+1})\, dx_{2n+1}\,.$$

This gives

$$\phi_T(t) = \int_{R^{2n}} \varphi_1(t, x)\, H_T(t, x)\, dx.$$

Next we set

$$\hat{\varphi}_1(t, u) = \int_{R^{2n}} \varphi_1(t, x) \exp\left(i\left(\sum_{k=1}^{n} u_k x_k\right)\right) d\bar{x}$$

and $\omega(t, u) = \exp(-i\,\frac{1}{2}(\sum_{k=1}^{n} \varphi_k(t)(u_{2k-1}^2 + u_{2k}^2)))$, and observe, that

$$\phi_T(t) = \int_{C_T} \hat{\varphi}_1(t, u)\, \omega(t, u)\, du$$

(for C_T cf. the begin of a)).

Therefore

$$|\phi_T(t)| \leqslant \int_{R^{2n}} |\hat{\varphi}_1(t, u)|\, du,$$

and since $\varphi_1(t, \bar{x})$ is C_c^∞ on $R^m \times R^{2n}$, the right hand side is under a bound M, not depending on $t \in R^m$.

6.4. The result proved below will be needed in 6.5. We denote by $L_n(x)$ the nth Laguerre polynomial (cf. [7], p. 93), and recall, that putting $\mathscr{L}_n(x) = \exp(\frac{1}{2}x)\, L_n(x)/n!$, the system $\{\mathscr{L}_n(x);\ n = 1, 2,...,\}$ is orthogonal and complete in $L^2([0, +\infty))$ (with respect to the Lebesgue measure). Let m be a fixed positive integer; we write $\mathscr{N} = \{N;\ N = (n_1, n_2, ..., n_m),\ n_k = \text{integer} \geqslant 0 \text{ for all } k\}$. Given N in $\mathscr{N}$, we set $\mathscr{L}_N(x) = \mathscr{L}_{n_1}(x_1)\, \mathscr{L}_{n_2}(x_2) \cdots \mathscr{L}_{n_m}(x_m)$. If $x = (x_1, x_2, ..., x_m) \in R^m$ is such, that $x_k \geqslant 0$ $(1 \leqslant k \leqslant m)$, and if α is any real number we denote by x^α the point $(x_1{}^\alpha, x_2{}^\alpha, ..., x_m{}^\alpha)$ of R^m. We denote by E^m the m-fold product of the half line $E = [0, +\infty)$ with itself. With these notations we have the following result. *Let $g(t, x)$ be a C_c^∞ function on $R^r \times E^m$ $(t = (t_1, t_2, ..., t_r') \in R^r)$, and let us put*

$$a_N(t) = \int_{E^m} g(t, x^{1/2})\, \mathscr{L}_N(x)\, dx, \qquad (N \in \mathscr{N}).$$

Then there is a positive constant C, not depending on t and N, such that

$$|a_N(t)| < CJ(N), \qquad \text{for all} \qquad N \in \mathscr{N},$$

where for $N = (n_1, n_2, ..., n_m)$ we put $J(N) = \prod_{k=1}^m (1 + n_k)^{-5/4}$.

a) To prove this statement, we write $\mathscr{N}_+ = \{N;\ N \in \mathscr{N},\ n_k > 0,\ 1 \leqslant k \leqslant m\}$, and observe, that it is enough to establish the above estimate for N in $\mathscr{N}_+$. In fact, let π be a subset of $1, 2,..., m$ and $n(\pi)$ the number of elements in π. Assuming $\pi = \{j_1, j_2, ..., j_{n(\pi)}\}$ $(1 \leqslant j_1 < j_2 < \cdots j_{n(\pi)} \leqslant m)$ and $x \in E^m$, we write $x' = (x_{j_1}, x_{j_2}, ..., x_{j_{n(\pi)}})$, $\bar{x} = (y_1, y_2, ..., y_m)$, where $y_j = x_j^{1/2}$ if $j \bar{\in} \pi$, and $y_j = x_j$ otherwise, $d\bar{x} = \prod_{j \bar{\in} \pi} dx_j$, and put

$$g_\pi(t, x') = \int_{E^{m-n(\pi)}} g(t, \bar{x}) \prod_{j \bar{\in} \pi} \exp(-\tfrac{1}{2}x_j)\, d\bar{x}.$$

Let us denote by $\mathscr{N}_\pi$ the subset $\{N;\ n_j = 0 \text{ if and only if } j \bar{\in} \pi\}$ of $\mathscr{N}$. Since $\mathscr{L}_0(x) \equiv e^{-\frac{1}{2}x}$ $(x \in R)$, we have for any N in $\mathscr{N}_\pi$

$$a_N(t) = \int_{E^{n(\pi)}} g_\pi(t, x')\, \mathscr{L}_N'(x')\, dx'$$

where $dx' = \prod_{j\in\pi} dx_j$ and $\mathscr{L}'_N(x') = \prod_{j\in\pi} \mathscr{L}_j(x_j)$. Therefore, having already proved our estimate for N in $\mathscr{N}_+$, it suffices to repeat the same reasoning for each fixed π; if C_π is the resulting constant, we can finally take $C = \sup C_\pi$.

b) We recall (cf. [7], p. 94), that $L_n(x)$ satisfies $xy'' + (1 - x)\,y' + ny = 0$, and therefore $D\mathscr{L}_n = n\mathscr{L}_n$ ($n = 0, 1, 2,...,$), where $D = -(d/dx)\,x(d/dx) + \frac{1}{2}(\frac{1}{2}x - 1)$. Let $g(x)$ and $f(x)$ be C_c^∞ and C^∞ respectively on E. We have

$$\int_E g(x^{1/2})(Df)(x)\,dx = \int_E D(g(x^{1/2}))f(x)\,dx.$$

But $D(g(x^{1/2})) = h(x^{1/2})/x^{1/2}$, where $h(x) = \frac{1}{2}(-\frac{1}{2}g'(x) - \frac{1}{2}xg''(x) + x(\frac{1}{2}x^2 - 1)\,g(x))$, along with $g(x)$, is C_c^∞ on E. From this we conclude, that if $g(t, x)$ is as above, $D_j = -(d/dx_j)\,x_j(d/dx_j) + \frac{1}{2}(\frac{1}{2}x_j - 1)$ and $\mathscr{D} = \prod_{j=1}^m D_j$, there exist a C_c^∞ function $h(t, x)$ on $R^r \times R^m$, such that we have for all f, which is C^∞ on E^m:

$$\int_{E^m} g(t, x^{1/2})(\mathscr{D}f)(x)\,dx = \int_{E^m} \frac{h(t, x^{1/2})}{(x_1x_2 \cdots x_m)^{1/2}} f(x)\,dx.$$

Since $(\mathscr{D}\mathscr{L}_N)(x) = (\prod_{k=1}^m n_k)\,\mathscr{L}_N(x)$, assuming $N \in \mathscr{N}_+$ and replacing f by $\mathscr{L}_N(x)$ we obtain

$$a_N(t) = \Big[1\Big/\Big(\prod_{k=1}^m n_k\Big)\Big]\, b_N(t)$$

where

$$b_N(t) = \int_{E^m} \frac{h(t, x^{1/2})}{(x_1x_2 \cdots x_m)^{1/2}} \mathscr{L}_N(x)\,dx.$$

Therefore, to complete our proof, it is enough to find a constant C_1, such that $|\,b_N(t)| < C_1 \prod_{k=1}^m n_k^{-1/4}$ for all $t \in R^r$ and $N \in \mathscr{N}_+$.

c) In the following C_2, C_3 etc. will stand for constants, not depending on t or N (whichever applies). We recall, that for $n = 0, 1, 2,...,$

$$\mathscr{L}_n(x) = \frac{e^{\frac{1}{2}x}}{n!} \int_E e^{-t}t^n J_0(2(tx))^{1/2}\,dt;$$

$J_0(x)$ is the Bessel function of order zero (cf. [19] p. 102; cf. p. 99 loc. cit. for the difference in notation, as compared with [7]). Therefore, writing $N! = n_1!n_2! \cdots n_m!$ ($N \in \mathscr{N}_+$) we have

$$b_N(t) = 2^m/N! \int_{E^m} e^{-(\tau_1+\tau_2+\cdots+\tau_m)}\tau_1^{n_1}\tau_2^{n_2} \cdots \tau_m^{n_m} F(t, \tau^{1/2})\,d\tau$$

where

$$F(t, \tau) = \int_{E^m} h(t, u) \prod_{k=1}^{m} J_0(2\tau_k u_k) \cdot du.$$

It is well known, that there exist a positive constant M, such that, for $x > 0$, $|J_0(x)| < M/(x)^{1/2}$. Therefore

$$|F(t, \tau)| < C_2(\tau_1 \cdot \tau_2 \cdots \tau_m)^{-1/2}$$

for all $t \in R^r$, and thus

$$|b_N(t)| < C_3 \prod_{k=1}^{m} (\Gamma(n_k + \tfrac{3}{4})/\Gamma(n_k + 1))$$

where Γ is the gamma function. But, since C_4 appropriately chosen, we have $\Gamma(n + 3/4)/\Gamma(n + 1) < C_4 . n^{-1/4}$ $(n = 1, 2,...)$, we obtain finally, that for all $t \in R^r$ and N

$$|b_N(t)| < C_1(n_1 n_2 \cdots n_m)^{-1/4},$$

which is the desired conclusion.

6.5. We now turn to the computation of the trace of the operator $T(\varphi)$ $(\varphi \in P(T)$; cf. 5.1), where T is an irreducible representation, as described in 6.2, corresponding to $\gamma = 1$.

a) For N in $\mathcal{N}$ (m replaced by n), let us write

$$|N| = n_1 + n_2 + \cdots + n_n \quad \text{and} \quad z^N = z_1^{n_1} z_2^{n_2} \cdots z_n^{n_m}$$
$$(z = (z_1, z_2, ..., z_n) \in C^n).$$

We put $f_N(z) = (2\pi)^{-\frac{1}{2}n} \cdot 2^{-\frac{1}{2}N} \cdot (N!)^{-1/2} \cdot z^N$ and observe, that the system $\{f_N(z); N \in \mathcal{N}\}$ is orthonormal and complete in $H(T)$. For $a = (t; v; u) \in G$ $(t = (t_1, t_2, ..., t_m) \in R^m$, $v = (v_1, v_2, ..., v_n) \in C^n$, $u \in R)$ and $N \in \mathcal{N}$ let us write $G_N(a) = (T(a)f_N, f_N)$. Putting for $v \in C$

$$F_n(v) = \int_C (z + v)^n \cdot \bar{z}^n \cdot \exp(-\tfrac{1}{2}z\bar{v}) \exp(-\tfrac{1}{2}|z|^2)\, dz,$$
$$(n = 1, 2, ...,)$$
$$H_n(v) = (1/2\pi) \cdot 2^{-n} \cdot (1/n!) \exp(-\tfrac{1}{4}|v|^2) F_n(v),$$

and writing $\omega_N(t) = \exp(-i(\sum_{k=1}^{n} n_k \varphi_k(t)))$, an easy computation shows, that we have

$$G_N(a) = \exp(i\lambda t) \exp(iu)\, \omega_N(t) \prod_{k=1}^{n} H_{n_k}(\varphi_k(t), v_k).$$

b) A simple verification yields, that

$$F_n(v) = \pi \cdot 2^{n+1} \cdot n! \left(\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{v^{2k}}{2^k \cdot k!} \right).$$

On the other hand (cf. [7], p. 93)

$$L_n(x) = n! \left(\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{x^k}{k!} \right)$$

and therefore $F_n(v) = \pi \cdot 2^{n+1} \cdot L_n(\frac{1}{2}| v |^2)$, and $H_n(v) = \mathscr{L}_n(\frac{1}{2}| v |^2)$. Writing for $v \in C^n : \frac{1}{2}| v |^2 = (\frac{1}{2}| v_1 |^2,..., \frac{1}{2}| v_n|^2) \in R^n$, we conclude, that $G_N(a) = \exp(i\lambda t) \exp(iu) \mathscr{L}_N(\frac{1}{2}| v |^2)$ $(N \in \mathscr{N})$.

c) Let $v_j = x_j + iy_j$; putting $dv_j = dx_j\, dy_j$ $(1 \leqslant j \leqslant n)$ and $dv = \prod_{k=1}^{n} dv_k$, one sees easily, that $da = dt\, dv\, du$ is the element of a biinvariant measure on G. Writing $a_N = \int_G \varphi(a)\, G_N(a)\, da$, we have

$$a_N = \int_{R^m} \left(\int_{E^n} H(t, r^{1/2})\, \mathscr{L}_N(r)\, dr \right) \omega_N(t)\, dt \qquad (N \in \mathscr{N})$$

where

$$H(t, r) = \exp(i\lambda t) \int_{I^n} \varphi(t;\, (2)^{1/2}\, re^{i\varphi};\, u) \cdot e^{iu}\, d\varphi\, du$$

with $C^n \ni re^{i\varphi} = (r_1e^{i\varphi_1}, r_2e^{i\varphi_2},..., r_ne^{i\varphi_n})$ and $I = [0, 2\pi]$. Let us set

$$b_N(t) = \int_{E^n} H(t, r^{1/2})\, \mathscr{L}_N(r)\, dr.$$

Since $H(t, r)$ is C_c^∞ on $R^m \times E^n$, by virtue of 6.4, with notations as loc. cit., we have $| b_N(t)| < C \cdot J(N)$ $(N \in \mathscr{N})$. For $\delta \geqslant 0$ we write $F(\delta) = \{t; t \in R^m, | \varphi_k(t)| > \delta, 1 \leqslant k \leqslant n\}$ and

$$a_N(\delta) = \int_{F(\delta)} \omega_N(t)\, b_N(t)\, dt.$$

There is a constant C_1, such that $| a_N(\delta)| < C_1 J(N)$ $(N \in \mathscr{N})$ and thus the series $\sum_{N \in \mathscr{N}} a_N(\delta)$ is absolutely convergent. Denoting its sum by $I(\delta)$, we have $\lim_{\delta \to 0} I(\delta) = I(0) = \mathrm{Tr}(T(\varphi))$, and hence, in particular, $T(\varphi)$ is of trace class.

d) For $\delta > 0$ fix, and $0 < \epsilon < 1$ we put

$$I(\epsilon; \delta) = \sum_{N \in \mathscr{N}} a_N(\delta)\, \epsilon^{|N|}.$$

We have $\lim_{\epsilon\to 1} I(\epsilon; \delta) = I(\delta)$. We recall now (cf. [7], p. 93), that for $0 \leqslant x$ and $z \in C$ with $|z| < 1$ we have

$$\sum_{n=0}^{\infty} \mathscr{L}_n(x)\, z^n = \left[\exp\left(-\left(\tfrac{1}{2} + \frac{xz}{1-z}\right)\right)\right] \Big/ (1-z).$$

From this we conclude first, that given $0 < \epsilon < 1$ and $M > 0$, there exist constants $C_2 > 0$ and η, $0 < \eta < 1$, such that

$$|\mathscr{L}_N(r)|\, \epsilon^{|N|} < C_2 \eta^{|N|}$$

for all $N \in \mathscr{N}$, provided $r_j \leqslant M$ ($r = (r_1, r_2, ..., r_n)$, $r_j \geqslant 0$, $1 \leqslant j \leqslant n$). Therefore, putting $z_j = \exp(-i\varphi_j(t))$ $(1 \leqslant j \leqslant n)$, we get

$$\begin{aligned}
I(\epsilon, \delta) &= \sum_{N\in\mathscr{N}} \int_{F(\delta)} \left(\int_{E^m} H(t, r^{1/2})\, \mathscr{L}_N(r)\, z^N\, dr\right) dt \\
&= \int_{F(\delta)} \left[\int_{E^m} \left(H(t, r^{1/2}) \left(\sum_{N\in\mathscr{N}} \mathscr{L}_N(r)\, z^N\right)\right) dr\right] dt \\
&= \int_{F(\delta)} \left[\int_{E^m} H(t, r^{1/2})\, K(t, r; \epsilon)\, dr\right] dt,
\end{aligned}$$

where

$$K(t, r; \epsilon) = \prod_{j=1}^{n} \left[\exp(-\tfrac{1}{2} r_j - (r_j z_j)/(1 - z_j))/(1 - z_j)\right].$$

We observe now, that since $\varphi \in P(T)$, the projection of the support of $\varphi(t; v; u)$ from $R^m \times C^n \times R^1$ onto R^m is a compact subset of $\{t;\, t \in R^m,\ |\varphi_j(t)| < 2\pi\}$. In this fashion, if δ is small enough, $\varphi(t; v; u) \neq 0$ implies $|\varphi_j(t)| < 2\pi - \delta$ $(1 \leqslant j \leqslant n)$. On the other hand, we have for all complex z satisfying $0 < |z| < 1$ and $0 < \delta < \arg(z) < 2\pi - \delta$, and a real r with $0 \leqslant r \leqslant M$:

$$|\exp(-(rz)/(1-z))/(1-z)| \leqslant \exp(M/\sin(\delta))/\sin(\delta).$$

Therefore, setting $K(t, r) = K(t, r; 1)$ we can conclude, that

$$I(\delta) = \lim_{\epsilon\to 1} I(\epsilon, \delta) = \int_{F(\delta)} \left(\int_{E^n} H(t, r^{1/2})\, K(t, r)\, dr\right) dt$$

whence, by expressing H in terms of φ (cf. c)) we get

$$I(\delta) = \int_{F(\delta)} e^{i\lambda t} \left(\int_{C^n\times R^1} \varphi(t; v; u)\, K(t, \tfrac{1}{2}|v|^2)\, e^{iu}\, dv\, du\right) dt.$$

e) Let us consider again the parametrization $a = (t; v; u)$ ($a \in G$; $t \in R^m$, $v \in C^n$, $u \in R$) of G introduced in 6.2. An easy verification shows, that the system of $m + 2n + 1$ real coordinates (τ, u, s) ($\tau \in R^m$, $u \in R^{2n}$, $s \in R^1$), connected with the members of the previous system through

$$\begin{aligned} t_j &= \tau_j \qquad (1 \leqslant j \leqslant m), \\ v_j &= x_j + iy_j = [(e^{i\varphi_j(\tau)} - 1)/i\varphi_j(\tau)](u_{2j-1} + iu_{2j}) \qquad (1 \leqslant j \leqslant n), \\ u &= s + \sum_{j=1}^{n} [(u_{2j-1}^2 + u_{2j}^2)/2(\varphi_j(\tau))^2] \operatorname{Im}(1 + i\varphi_j(\tau) - e^{i\varphi_j(\tau)}), \end{aligned}$$

defines a system of canonical coordinates of the first kind for G. Putting $d\bar{u} = \prod_{j=1}^{n} du_j$, we get

$$da = dt\, dv\, du = \prod_{j=1}^{n} |(e^{i\varphi_j(\tau)} - 1)/i\varphi_j(\tau)|^2\, d\tau\, d\bar{u}\, ds,$$

and

$$\begin{aligned} K(t, \tfrac{1}{2} \mid v \mid^2)\, du = {} & \Big(\prod_{j=1}^{n} [i\varphi_j(\tau)]^{-1}\Big)\Big(\prod_{j=1}^{n} [(e^{i\varphi_j(\tau)} - 1)/i\varphi_j(\tau)]\Big) \\ & \times \exp\Big[\sum_{k=1}^{n} i\, \frac{u_{2k-1}^2 + u_{2k}^2}{2\varphi_k(\tau)}\Big]\, d\tau\, d\bar{u}\, ds. \end{aligned}$$

Substituting this in the final formula of d), we obtain for $I(\delta)$ the formula

$$\begin{aligned} \int_{F(\delta)} e^{i\lambda\tau} \prod_{j=1}^{n} [i\varphi_j(\tau)]^{-1} \Big[\int_{R^{2n+1}} \varphi(\tau, u, s)\Big[\prod_{j=1}^{n} \Big(\frac{e^{i\varphi_j(\tau)} - 1}{2\varphi_j(\tau)}\Big)\Big] \\ \times \exp\Big(\sum_{k=1}^{n} i\, \frac{(u_{2k-1}^2 + u_{2k}^2)}{2\varphi_k(\tau)}\Big)\, e^{is}\, d\bar{u}\, ds\Big]\, d\tau. \end{aligned}$$

Let us replace now in the expression of $\psi(t)$ (cf. the begin of 6.3) the C_c^∞ function $\varphi(t, x)$ on $R^m \times R^{2n}$ by the function of the same kind $\omega(t, x) \equiv \varphi(t, x) \prod_{j=1}^{n} [(e^{i\varphi_j(\tau)} - 1)/i\varphi_j(\tau)]$. Comparing the resulting expression with the right hand side of the last formula, we conclude, that

$$I(\delta) = \int_{F(\delta)} \psi(\tau)\, d\tau.$$

We know (cf. 6.3), that $|\psi(\tau)|$ is bounded, and therefore

$$\mathrm{Tr}(T(\varphi)) = \lim_{\delta\to 0} I(\delta) = \int_{R^m} \psi(\tau)\, d\tau = \int_O \hat{\omega}(l')\, dv$$

by virtue of the main result of 6.3, where dv is the invariant measure $d\bar{y}/(2\pi)^n$ on O. Recalling the statement of the Theorem in 5.1, we see, that this is the relation to be established. In fact, with notations as loc. cit., it is enough to take for $\mathscr{F}_2$ the system of roots $\{i\varphi_k;\ 1 \leqslant k \leqslant n\}$ of $\mathscr{L}$, and for $\mathscr{F}_1$ the empty set.

One shows easily, that if $\gamma > 0$ (but not necessarily 1), the above considerations remain in force with the difference, that for dv we have to take $d\bar{y}/(2\pi\gamma)^n$. If $\gamma < 0$, then using the realization, given at the end of 6.2, for the corresponding representation, one sees at once, that $dv = d\bar{y}/(2\pi|\gamma|)^n$, and that $\mathscr{F}_2 = \{-i\varphi_k;\ 1 \leqslant k \leqslant n\}$.

6.6. Let us prove finally, that the canonical measure dv obtained above is the same, as what we get by using the algorithm of 5.6. To this end, let us observe first, that the Haar measure da and the linear measure $dl = d\tau\, d\bar{u}\, ds$ on, used above, satisfy $da/dl\,|_0 = 1$. Therefore what we have to prove is that dv equals the Kostant measure of 0 divided by $C(d) = 2^n \cdot n!(2\pi)^n$ (cf. for all this 5.6). We know (cf. 6.2), that R is spanned by the system $\{e_{2n+1}, a_k;\ 1 \leqslant k \leqslant n\}$, and therefore the system $\{e_k;\ 1 \leqslant k \leqslant 2n\}$ is a supplementary base to R in $\mathscr{L}$. Let us form $g(T)$ as in 5.6. a); by taking into account, that for $i < j : ([e_i, e_j], f) = \gamma$ if $i = 2r - 1$ and $j = 2r$, and zero otherwise, we conclude, that at f

$$\omega = 2\gamma \left(\sum_{j=1}^{n} dt_{2j-1} \wedge dt_{2j}\right)$$

and thus the Kostant measure at f equals $2^n \cdot |\gamma|^n \cdot n! \cdot dt_1\, dt_2 \cdots dt_{2n}$. On the other hand, from

$$\rho([g(T)]^{-1}) f = \gamma e'_{2n+1} + \sum_{k=1}^{2n} y_k e'_k + \sum_{j=1}^{m} \tau_j(\bar{y})\, a'_j$$

(cf. 6.1) we obtain readily $d\bar{y} = \gamma^{2n}\, dt$ implying, that at f, $dv = d\bar{y}/(2|\gamma|\pi)^n = (|\gamma|^n/(2\pi)^n)\, dt$, that is, the Kostant measure divided by $2^n \cdot n! \cdot (2\pi)^n$, which is the desired conclusion.

References

1. Auslander, L. and Kostant, B., Quantization and representations of solvable Lie groups, *Bull. Am. Math. Soc.* **73** (1967), 692–695.
2. Auslander, L. and Moore, C. C., Unitary representations of solvable Lie groups, Memoirs of the Am. Math. Soc. No. 62, 1966.
3. Blattner, R. J., On induced representations, *Am. J. Math.* **83** (1961), 79–98.
4. Chevalley, C., A new relationship between matrices, *Am. J. Math.* **65** (1943), 521–531.
5. Chevalley, C., Théorie des groupes de Lie, t. II, Groupes algébriques, *Act. Sci. Ind.* No. 1152, Paris, Hermann, 1951.
6. Chevalley, C. and Hsio-Fu Tuan, On algebraic Lie algebras, *Proc. Nat. Acad. Sci.* **31** (1945), 195–196.
7. Courant, R. and Hilbert, D., "Methods of mathematical physics," vol. 1, Interscience, 1953.
8. Dixmier, J., Sur les représentations unitaires des groupes de Lie algébriques, *Ann. Inst. Fourier* **7** (1957), 315–328.
9. Dixmier, J., L'application exponentielle dans les groupes de Lie résolubles, *Bull. Soc. Math. France* **85** (1957), 113–121.
10. Dixmier, J., Représentations induites holomorphes des groupes résolubles algébriques, *Bull. Soc. Math. France* **94** (1966), 181–206.
11. Goto, M., On algebraic Lie algebras, *J. Math. Soc. Japan* **1** (1948), 29–45.
12. Mackey, G. W., Induced representations of a locally compact groups I, *Ann. Math.* **55** (1952), 101–139.
13. Matsushima, Y., On algebraic Lie groups and algebras, *J. Math. Soc. Japan* **1** (1948), 46–57.
14. Neumann, J. v., Die Eindeutigkeit der Schrödingerschen Operatoren, *Math. Ann.* **104** (1931), 570–578.
15. Pukanszky, L., "Leçons sur les représentations des groupes," Paris, Dunod, 1967.
16. Pukanszky, L., On the characters and the Plancherel formula of nilpotent groups, *J. Funct. An.* **1** (1967), 255–280.
17. Pukanszky, L., On the unitary representations of exponential groups, *J. Funct. An.* **2** (1968), 73–113.
18. Seidenberg, A., A new decision method for elementary algebra, *Ann. Math.* **60** (1954), 365–374.
19. Szegö, G., Orthogonal polynomials, *Am. Math. Soc. Coll. Publ.* v. XXIII.
20. Weil, A., "L'intégration dans les groupes topologiques et ses applications," Paris, Hermann, 1953.